

CHARACTER SHEAVES ON CERTAIN SPHERICAL VARIETIES

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ABSTRACT. We study a class of perverse sheaves on some spherical varieties which include the strata of the De Concini-Procesi completion of a symmetric variety. This is a generalization of the theory of (parabolic) character sheaves.

INTRODUCTION

0.1. Let G be a connected, reductive algebraic group over an algebraically closed field \mathbf{k} . In [L3] and [L4], Lusztig introduced the $(G \times G)$ -varieties $Z_{J,y,D}$ and a class of G -equivariant simple perverse sheaves on $Z_{J,y,D}$ which are called “(parabolic) character sheaves”. (The precise definition of $Z_{J,y,D}$ can be found in 1.2 below). The varieties $Z_{J,y,D}$ include as a special case the group G itself. In this special case, the “(parabolic) character sheaves” on G are just the usual character sheaves on G introduced by Lusztig in [L1]. The varieties also include more or less as a special case the boundary pieces of the De Concini-Procesi compactification of G (where G is adjoint).

0.2. We now review [L3] and [L4] in more detail.

For $Z_{J,y,D}$, there exists a finite partition into some smooth, G -stable subvarieties which we call G -stable pieces. This partition is based on some combinatorial result of Bédard (see 2.2). The G -orbits on each piece are in one-to-one correspondence with the “twisted” conjugacy classes of a certain (smaller) reductive subgroup L . Furthermore, there is a natural equivalence between the bounded derived category of G -equivariant, constructible sheaves on that piece and the boundary derived categories of L -equivariant (for the twisted conjugate action), constructible sheaves on L .

To each character sheaf on L , one can associate a G -equivariant simple perverse sheaf on the G -stable piece and call it a character sheaf on the G -stable piece. This provides the “local picture” of the theory of parabolic character sheaves. By imitating the definition of character sheaves on the group, one can obtain certain simple perverse sheaves on $Z_{J,y,D}$ and call them character sheaves on $Z_{J,y,D}$. This provides the “global picture”.

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Lusztig proved the following property:

Let i be the inclusion of a G -stable piece to $Z_{J,y,D}$, then

- (1) for any character sheaf C on $Z_{J,y,D}$, any perverse constituent of $i^*(C)$ is a character sheaf on that piece;
- (2) for any character sheaf C on that piece, any perverse constituent of $i_!(C)$ is a character sheaf on $Z_{J,y,D}$.

As a consequence, the character sheaves on $Z_{J,y,D}$ are just the perverse extensions to $Z_{J,y,D}$ of the character sheaves on G -stable pieces.

These results were also proved later in [Sp2] and [H2] in some different way. In all these proofs, some inductive methods based on Bédard's result were used. For more details, see the introduction of [H2].

0.3. Let τ be an involution on G and G^τ be the τ -fixed point subgroup. In this paper, we study (under a mild assumption on the characteristic of \mathbf{k}) a class of G^τ -equivariant simple perverse sheaves on varieties $X_{J,\tau}$ which we call “(parabolic) character sheaves”. The varieties $X_{J,\tau}$ are defined in 3.5 and include the varieties $Z_{J,y,D}$ as some special cases. They also include as a special case the strata of the De Concini-Procesi compactification of the symmetric variety G/G^τ (hence the symmetric variety G/G^τ itself). For more details, see 3.30 and 3.31.

0.4. To achieve this goal, the first thing we need to do is to find a partition of $X_{J,\tau}$ which is analogous to the partition of $Z_{J,y,D}$ into G -stable pieces. We call it the partition of $X_{J,\tau}$ into G^τ -stable pieces. However, there is no results in our general setting that is analogous to Bédard's result. Hence we need to find a different approach.

The idea is to relate the variety $X_{J,\tau}$ to certain $Z_{J,y,D}$. In the special case where $X_{J,\tau} = G/G^\tau$, we can identify G/G^τ with the identity component of $G^{\iota\circ\tau}$, where ι is the inverse map on G (see [Gi, 3.3.0]). This result can be easily generalized. Namely, we can identify $X_{J,\tau}$ with certain irreducible component of $Z_{J,y,D}^{\iota\circ\tau}$, where ι is the “inverse” map on $Z_{J,y,D}$ (see 3.5).

Moreover, $\iota \circ \tau$ maps an G -stable piece in $Z_{J,y,D}$ to another G -stable piece. This is what we will show in section 2. Although this result is not needed to establish the partition of $X_{J,\tau}$, it serves as motivation for it. Namely, it suggests that the G^τ -stable pieces might be the irreducible components of the intersections of $X_{J,\tau}$ with the G -stable pieces in $Z_{J,y,D}$.

In fact, this is the right definition. (Certainly we need to show that each connected component of the intersection is irreducible and we need to know when the intersection is nonempty and what are the components, etc.) Actually, in 3.21 we will define the G^τ -stable pieces in an equivalent way that doesn't involve the G -stable pieces in $Z_{J,y,D}$.

In section 3, we will also prove some results on the structure of G^τ -stable pieces (see 3.26 and 3.28) and show that the G^τ -orbits on each G^τ -stable piece are in one-to-one correspondence with the L^τ -orbits

on L/L^{τ_2} , where L is a (smaller) reductive group and τ_1, τ_2 are two involutions on L (see 3.29).

0.5. Based on these results, we can establish a natural equivalence between the bounded derived category of G^τ -equivariant, constructible sheaves on that piece and the boundary derived categories of L^{τ_1} -equivariant, constructible sheaves on L/L^{τ_2} (see 4.14).

Hence we obtain the “local picture” in the same way as in 0.2. The “global picture” is obtained by imitating Ginsburg’s definition of character sheaves on symmetric varieties in [Gi]. There is also a characterization of character sheaves using Ginsburg’s Harish-Chandra functor. This characterization will play an essential role in our proofs of the properties analogous to those in 0.2.

Finally, in section 5, we generalize Lusztig’s functors $e_J^{J'}$ and $f_J^{J'}$ and prove some properties.

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1. THE G -STABLE PIECES

1.1. Let B be a Borel subgroup of G , B^- be the opposite Borel subgroup and $T = B \cap B^-$. Let $(\alpha_i)_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by s_i the corresponding simple reflection. For any element w in the Weyl group $W = N(T)/T$, we use the same symbol w for a representative of w in $N(T)$. We denote by $\text{supp}(w) \subset I$ the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of w .

For $J \subset I$, let W_J be the subgroup of W generated by J and

$$W^J = \{w \in W; w = \min(wW_J)\}, \quad {}^JW = \{w \in W; w = \min(W_Jw)\}.$$

For $J, K \subset I$, we write ${}^JW^K$ for ${}^JW \cap W^K$.

Let Φ^+ be the set of positive roots in G and Φ_J be the set of roots in L_J . Set $\Phi_J^+ = \Phi_J \cap \Phi^+$. Let $P_J \supset B$ be the standard parabolic subgroup defined by J and \mathcal{P}_J be the set of parabolic subgroups conjugate to P_J . Let $P_J^- \supset B^-$ be the opposite of P_J . Set $L_J = P_J \cap P_J^-$. Then L_J is a common Levi subgroup of P_J and P_J^- . Let $\pi_J : P_J \rightarrow L_J$ be the projection map.

For any parabolic subgroup P , we denote by U_P its unipotent radical. We simply write U for U_B . For $J \subset I$, we simply write U_J for $U \cap L_J$ and B_J for $B \cap L_J$.

For $J, K \subset I$, $P \in \mathcal{P}_J$, $Q \in \mathcal{P}_K$ and $u \in {}^JW^K$, we write $\text{pos}(P, Q) = u$ if there exists $g \in G$, such that ${}^gP = P_J$ and ${}^gQ = {}^uP_K$.

For any closed subgroup H of G , we denote by $\text{Lie}(H)$ the corresponding Lie subalgebra and denote by H_Δ the image of the diagonal

embedding of H in $G \times G$. For any subgroup H and $g \in G$, we write gH for gHg^{-1} .

For each root α , we denote by u_α the one-dimensional subspace in $\text{Lie}(G)$ that corresponds to α .

If θ be an automorphism on G with $\theta(T) = T$, then θ induces a bijection on the set of roots and an automorphism on W . By abuse of notation, we use the same symbol θ for the induced maps. We also use the same symbol θ for the induced map on $\text{Lie}(G)$.

For a group, we use ι for the inverse map. For an automorphism f on a variety X , we write X^f for the fixed point set.

1.2. Let \hat{G} be an algebraic group with identity component G and D be a fixed irreducible component of \hat{G} . By the conjugation of Borel subgroups and maximal tori we may find an element $g_D \in D$ such that g_D normalizes B and T . Set $\delta = \text{Ad}(g_D)$.

Let $J, J' \subset I$ and $y \in {}^{J'}W^{\delta(J)}$ be such that $y\delta(J) = J'$. For $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_{J'}$, define $A_{y,D}(P, P') = \{g \in D \mid \text{pos}(P', {}^gP) = y\}$. Define

$$Z_{J,y,D} = \{(P, P', \gamma); P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in U_{P'} \setminus A_{y,D}(P, P')/U_P\}$$

with $G \times G$ -action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1\gamma g_2^{-1})$.

By [L4, 8.9], $A_{y,D}(P, P')$ is a single P', P double coset. Thus $G \times G$ acts transitively on $Z_{J,y,D}$. Set

$$h_{J,y,D} = (P_J, {}^{\dot{y}^{-1}}P_{J'}, U_{{}^{\dot{y}^{-1}}P_{J'}} g_D U_{P_J}) \in Z_{J,y,D}.$$

We call $h_{J,y,D}$ the base point for the $G \times G$ -action on $Z_{J,y,D}$. Now we may identify $Z_{J,y,D}$ with $(G \times G)_{{}^{\dot{y}^{-1}}P_{J'} \times P_J} g_D L_J$ where ${}^{\dot{y}^{-1}}P_{J'} \times P_J$ acts on $G \times G$ on the right and acts on $g_D L_J$ by $(p, p') \cdot g_D l = \bar{p} g_D l \pi_J(p')^{-1}$ for $p \in {}^{\dot{y}^{-1}}P_{J'}$, $p' \in P_J$ and $l \in L_J$. Here \bar{p} is the image of p under the map ${}^{\dot{y}^{-1}}P_{J'} \rightarrow {}^{\dot{y}^{-1}}P_{J'}/U_{{}^{\dot{y}^{-1}}P_{J'}} \cong L_{\delta(J)}$.

1.3. For $w \in W^{\delta(J)}$, set $I(J, w, \delta) = \max\{K \subset J'; w\delta(K) = K\}$ and

$$Z_{J,y,D;w} = G_\Delta(Bw, B)h_{J,y,D}.$$

The varieties $Z_{J,y,D;w}$ are called the G -stable pieces in $Z_{J,y,D}$. They were introduced by Lusztig in [L4].

The following properties can be found in [L4, section 8] and [H1, section 1].

- (1) $Z_{J,y,D} = \bigsqcup_{w \in W^{\delta(J)}} Z_{J,y,D;w}$.
- (2) The map $G \times (P_{I(J,w,\delta)}w, P_{I(J,w,\delta)})h_{J,y,D} \rightarrow Z_{J,y,D}$ defined by $(g, z) \mapsto (g, g) \cdot z$ induces an isomorphism

$$Z_{J,y,D;w} \cong G \times_{P_{I(J,w,\delta)}} (P_{I(J,w,\delta)}w, P_{I(J,w,\delta)})h_{J,y,D},$$

where the group $P_{I(J,w,\delta)}$ acts on the right on G and acts diagonally on $(P_{I(J,w,\delta)}w, P_{I(J,w,\delta)})h_{J,y,D}$.

(3) The map

$$(U_{P_{I(J,w,\delta)}} \cap {}^{wy^{-1}}(U_J U_{P_J^-})) \times L_{I(J,w,\delta)} w \rightarrow (P_{I(J,w,\delta)} w, P_{I(J,w,\delta)}) h_{J,y,D}$$

defined by $(u, z) \mapsto (uz, 1) \cdot h_{J,y,D}$ is an isomorphism.

(4) The map $L_{I(J,w,\delta)} w \rightarrow Z_{J,y,D}$ defined by $z \mapsto (z, 1) \cdot h_{J,y,D}$ induces a bijection from the set of $L_{I(J,w,\delta)}$ -orbits on $L_{I(J,w,\delta)} w g_D$ to the set of G -orbits on $Z_{J,y,D;w}$.

2. THE INVERSE MAP AND THE G -STABLE PIECES

2.1. Let D be a connected component of \hat{G} . For the connected component D^{-1} of \hat{G} , we choose $g_{D^{-1}} = g_D^{-1}$. As in [L2, 28.19], we define the map

$$\partial : Z_{J,y,D} \rightarrow Z_{J',y^{-1},D^{-1}}$$

by $\partial(P, Q, \gamma) = (Q, P, \gamma^{-1})$. We call $\partial : Z_{J,y,D} \rightarrow Z_{J',y^{-1},D^{-1}}$ the inverse map.

If $J = I$ and $y = 1$, then $Z_{J,y,D} = D$ and ∂ is just the restriction to D of the inverse map on \hat{G} .

In Proposition 2.5, we will show that ∂ maps the G -stable pieces in $Z_{J,1,D}$ to the G -stable pieces in $Z_{\delta(J),1,D^{-1}}$.

2.2. In this subsection, we reformulate Bédard's description of $W^{\delta(J)}$. The description below is slightly different from [L3, 2.2]. (In fact, the sequence $(J_n, w_n)_{n \geq 0}$ below corresponds to the sequence $(J_n, w_n^{-1})_{n \geq 0}$ in *loc.cit.*)

Let $\mathcal{T}(J, \delta)$ be the set of all sequences $(J_n, w_n)_{n \geq 0}$ where $J_n \subset I$ and $w_n \in W$ such that

- (1) $J_0 = J$,
- (2) $J_n = J_{n-1} \cap \delta^{-1}(w_{n-1}^{-1} J_{n-1})$ for $n \geq 1$,
- (3) $w_n \in J_n W^{\delta(J_n)}$ and $w_n \in W_{J_{n-1}} w_{n-1}$ for $n \geq 1$.

For each sequence $(J_n, w_n) \in \mathcal{T}(J, \delta)$, we have that $J_m = J_{m+1} = \dots$ and $w_m = w_{m+1} = \dots$ for $m \gg 0$. By [L3, 2.4 & 2.5], $w_m \in W^{\delta(J)}$ for all $m \geq 0$ and the map $\mathcal{T}(J, \delta) \rightarrow W^{\delta(J)}$ defined by $(J_n, w_n)_{n \geq 0} \mapsto w_m$ for $m \gg 0$ is a bijection. Moreover, by [H1, 1.4], $J_m = I(J, w_m, \delta)$ for $m \gg 0$.

2.3. To $(J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$, we associate a sequence $(K_n, v_n)_{n \geq 0}$ with $K_n \subset I$ and $v_n \in W$. We set

$$K_0 = \delta(J), \quad v_0 = w_0^{-1}.$$

Assume that $n \geq 1$ and that K_m, v_m are already defined for $m < n$. Let

$$K_n = K_{n-1} \cap \delta(v_{n-1}^{-1} K_{n-1}),$$

$$v_n = (\delta^n(v_0) \delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1} \delta^n(w_n^{-1}) (\delta^{n-1}(v_0) \delta^{n-2}(v_1) \cdots v_{n-1}).$$

This completes the inductive definition.

Lemma 2.4. *We have that $(K_n, v_n)_{n \geq 0} \in \mathcal{T}(\delta(J), \delta^{-1})$.*

Proof. We show by induction on $n \geq 0$ that

$$(a) \quad K_n = (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^{n+1}(J_n).$$

For $n = 0$, (a) is clear. Assume now that $n > 0$ and that the statement holds when n is replaced by $n - 1$. Then

$$\begin{aligned} K_{n-1} &= (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-2}))^{-1}\delta^n(J_{n-1}) \\ &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(w_{n-1})^{-1}\delta^{n+1}(J_n) \end{aligned}$$

and

$$\begin{aligned} K_n &= K_{n-1} \cap \delta(v_{n-1}^{-1}K_{n-1}) \\ &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(w_{n-1}^{-1}J_{n-1}) \\ &\quad \cap (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^{n+1}(J_{n-1}) \\ &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^{n+1}(\delta^{-1}(w_{n-1}^{-1}J_{n-1}) \cap J_{n-1}) \\ &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^{n+1}(J_n). \end{aligned}$$

(a) is proved.

By definition, $v_0 \in {}^{K_0}W^{\delta^{-1}(K_0)}$.

Let $n \geq 1$. By definition, $w_n \in {}^{J_n}W^{\delta(J_n)}$, $w_{n-1} \in {}^{J_{n-1}}W^{\delta(J_{n-1})}$ and $w_n w_{n-1}^{-1} \in W_{J_{n-1}}$. Thus

$$(b) \quad \begin{aligned} w_n \delta(\Phi_{J_n}^+) &= w_n \delta(\Phi_{J_n}) \cap \Phi^+ \subset w_n w_{n-1}^{-1}(\Phi_{J_{n-1}}) \cap \Phi^+ \\ &= \Phi_{J_{n-1}} \cap \Phi^+ = \Phi_{J_{n-1}}^+, \end{aligned}$$

$$(c) \quad \begin{aligned} w_n^{-1}(\Phi_{J_n}^+) &= w_n^{-1}(\Phi_{J_n}) \cap \Phi^+ \subset w_n^{-1}(\Phi_{J_{n-1}}) \cap \Phi^+ \\ &= w_{n-1}^{-1}(\Phi_{J_{n-1}}) \cap \Phi^+ = w_{n-1}^{-1}(\Phi_{J_{n-1}}^+). \end{aligned}$$

From (a) and the definition of v_n , we deduce that

$$(d) \quad v_n \delta^{-1}(\Phi_{K_n}^+) = (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(w_n)^{-1}\delta^n(\Phi_{J_n}^+),$$

$$(e) \quad v_n^{-1}(\Phi_{K_n}^+) = (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(w_n)\delta^{n+1}(\Phi_{J_n}^+).$$

From (c) and (d), we see that

$$\begin{aligned} v_n \delta^{-1}(\Phi_{K_n}^+) &\subset (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(w_{n-1})^{-1}\delta^n(\Phi_{J_{n-1}}^+) \\ &= (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-2}))^{-1}\delta^n(\Phi_{J_{n-1}}^+) = \Phi_{K_{n-1}}^+. \end{aligned}$$

From (b) and (e), we see that

$$v_n^{-1}(\Phi_{K_n}^+) \subset (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-1}))^{-1}\delta^n(\Phi_{J_{n-1}}^+) = v_{n-1}^{-1}\Phi_{K_{n-1}}^+.$$

Using induction method, we deduce $v_n^{-1}(\Phi_{K_n}^+) \subset \Phi^+$ for all $n \geq 0$.

Therefore $v_n \in {}^{K_n}W^{\delta^{-1}(K_n)}$.

Let $n \geq 1$. By definition,

$$\begin{aligned} \delta(v_{n-1}) &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta^2(v_{n-2}))^{-1} \delta^n(w_{n-1})^{-1} \\ &\quad (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-2})). \end{aligned}$$

Thus

$$\begin{aligned} \text{(f)} \quad v_n &= (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots \delta(v_{n-2}))^{-1} \delta^n(w_{n-1}w_n^{-1}) \\ &\quad (\delta^{n-1}(v_0)\delta^{n-2}(v_1) \cdots v_{n-1}). \end{aligned}$$

Notice that $w_{n-1}w_n^{-1} \in W_{J_{n-1}}$. By (a), $v_n \in W_{K_{n-1}}v_{n-1}$. \square

Proposition 2.5. *Define a map $\epsilon_{J,\delta} : W^{\delta(J)} \rightarrow W^J$ by sending the element $w \in W^{\delta(J)}$ corresponding to $(J_n, w_n)_{n \geq 0}$ to the element in W^J corresponding to $(K_n, v_n)_{n \geq 0}$. Then $\partial(Z_{J,1,D;w}) = Z_{\delta(J),1,D^{-1};\epsilon_{J,\delta}(w)}$.*

Proof. For $n \geq 0$, set

$$\begin{aligned} x_n &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1} \delta^{n+1}(w_n w^{-1}) \\ &\quad (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1})). \end{aligned}$$

Since $w_n w^{-1} \in W_{J_n}$, then $x_n \in W_{K_n}$. Therefore $l(x_n v_n) = l(x_n) + l(v_n)$ and $l(v_n \delta^{-1}(x_n)) = l(v_n) + l(x_n)$. By 2.4(f),

$$\begin{aligned} x_n v_n &= (\delta^n(v_0)\delta^{n-1}(v_1) \cdots \delta(v_{n-1}))^{-1} \delta^{n+1}(w_n w^{-1}) (\delta^n(v_0)\delta^{n-1}(v_1) \cdots v_n) \\ &= v_{n+1} (\delta^n(v_0)\delta^{n-1}(v_1) \cdots v_n)^{-1} \delta^{n+1}(w_{n+1} w^{-1}) (\delta^n(v_0)\delta^{n-1}(v_1) \cdots v_n) \\ &= v_{n+1} \delta^{-1}(x_{n+1}). \end{aligned}$$

By definition,

$$\begin{aligned} \partial(Z_{J,1,D;w}) &= \partial(G_\Delta(Bw, 1) \cdot h_{J,1,D}) = G_\Delta(1, Bw) \cdot h_{\delta(J),1,D^{-1}} \\ &= G_\Delta(w^{-1}B, 1) \cdot h_{\delta(J),1,D^{-1}} = G_\Delta B_\Delta(w^{-1}B, 1) \cdot h_{\delta(J),1,D^{-1}} \\ &= G_\Delta(Bv_0 \delta^{-1}(x_0)B, 1) \cdot h_{\delta(J),1,D^{-1}}. \end{aligned}$$

For $n \geq 0$,

$$\begin{aligned} G_\Delta(Bv_n \delta^{-1}(x_n)B, 1) \cdot h_{\delta(J),1,D^{-1}} &= G_\Delta(Bv_n B \delta^{-1}(x_n)B, 1) \cdot h_{\delta(J),1,D^{-1}} \\ &= G_\Delta(Bv_n B, Bx_n^{-1}B) \cdot h_{\delta(J),1,D^{-1}} = G_\Delta(Bx_n B v_n B, 1) \cdot h_{\delta(J),1,D^{-1}} \\ &= G_\Delta(Bv_{n+1} \delta^{-1}(x_{n+1})B, 1) \cdot h_{\delta(J),1,D^{-1}}. \end{aligned}$$

Therefore $\partial(Z_{J,1,D;w}) = G_\Delta(Bv_n \delta^{-1}(x_n), 1) \cdot h_{\delta(J),1,D^{-1}}$ for all $n \geq 0$. In particular,

$$\partial(Z_{J,1,D;w}) = G_\Delta(B\epsilon_{J,\delta}(w), B) \cdot h_{\delta(J),1,D^{-1}} = Z_{\delta(J),1,D^{-1};\epsilon_{J,\delta}(w)}.$$

\square

Notice that $\partial \circ \partial(Z_{J,1,D;w}) = Z_{J,1,D;w}$. We have the following consequence.

Corollary 2.6. *The map $\epsilon_{\delta(J),\delta^{-1}} \circ \epsilon_{J,\delta} : W^{\delta(J)} \rightarrow W^{\delta(J)}$ is the identity map.*

The following corollary gives another characterization of the map $\epsilon_{J,\delta}$.

Corollary 2.7. *For each $w \in W^{\delta(J)}$, there exists a unique element in W^J which is of the form $\delta(x)^{-1}w^{-1}x$ for some $x \in W_J$. This element is just $\epsilon_{J,\delta}(w)$.*

Proof. The existence of the element follows from the proof of Proposition 2.5. We prove the uniqueness. Assume that $\delta(x)^{-1}w^{-1}x \in W^J$ for some $x \in W_J$. Then

$$(\delta(x)^{-1}w^{-1}x, 1) \cdot h_{\delta(J),1,D^{-1}} \in Z_{\delta(J),1,D^{-1};\delta(x)^{-1}w^{-1}x}.$$

On the other hand,

$$\begin{aligned} (\delta(x)^{-1}w^{-1}x, 1) \cdot h_{\delta(J),1,D^{-1}} &= (\delta(x)^{-1}, \delta(x)^{-1})(w^{-1}x, \delta(x)) \cdot h_{\delta(J),1,D^{-1}} \\ &\in G_{\Delta}(w^{-1}T, 1) \cdot h_{\delta(J),1,D^{-1}} \subset \partial(Z_{J,1,D;w}) \\ &= Z_{\delta(J),1,D^{-1};\epsilon_{J,\delta}(w)}. \end{aligned}$$

Hence $\delta(x)^{-1}w^{-1}x = \epsilon_{J,\delta}(w)$. \square

In general, ∂ doesn't map a G -stable piece in $Z_{J,y,D}$ to a G -stable piece in $Z_{J',y^{-1},D^{-1}}$. However, we have a modified version which will be stated in the end of this section.

Lemma 2.8. *Let $x \in W$ and L be a Levi of some parabolic subgroup of L_J with ${}^{xgD}L = L$. Then there exists $w \in W^{\delta(J)}$, such that*

$$(Lx, 1) \cdot h_{J,y,D} \subset Z_{J,y,D;w}.$$

Proof. We define by induction on n a sequence $(J_n, w_n, u_n)_{n \geq 0}$ as follows.

We write x as $a\delta(b)$ for $a \in W^{\delta(J)}$ and $b \in W_J$ and set

$$J_0 = J, w_0 = \min(W_J a), u_0 = baw_0^{-1}.$$

Assume that $n > 0$ and that $J_{n-1}, w_{n-1}, u_{n-1}$ are defined. Let

$$J_n = J_{n-1} \cap \delta^{-1}(w_{n-1}^{-1}J_{n-1}).$$

Write $u_{n-1}w_{n-1}$ as $w'_n\delta(u'_n)$ for $w'_n \in W^{\delta(J_n)}$ and $u'_n \in W_{J_n}$. Set

$$w_n = \min(W_{J_n}w'_n), u_n = u'_nw'_nw_n^{-1}.$$

This completes the inductive definition.

We show that

$$(a) \quad w_n \in {}^{J_n}W^{\delta(J_n)} \quad \text{for } n \geq 0.$$

Let $y = \min(W_{J_n} w_n W_{\delta(J_n)})$. Now $w'_n \in W_{J_n} y W_{\delta(J_n)}$ and $w'_n \in W^{\delta(J_n)}$. By [L3, 2.1(b)], $w'_n \in W_{J_n} y$. Notice that $w'_n \in W_{J_n} w_n$ and $w_n, y \in {}^{J_n}W$. Then $w_n = y$ and (a) follows.

$$(b) \quad w_n \in W_{J_{n-1}} w_{n-1} \quad \text{for } n \geq 1.$$

By definition, $w_n w_{n-1}^{-1} \in W_{J_n} w'_n w_{n-1}^{-1} \subset W_{J_n} u_{n-1} w_{n-1} W_{\delta(J_n)} w_{n-1}^{-1}$. Since $u_{n-1} \in W_{J_{n-1}}$ and $J_n, w_{n-1} \delta(J_n) \subset J_{n-1}$, we have $w_n w_{n-1}^{-1} \in W_{J_{n-1}}$ and (b) follows.

$$(c) \quad (J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta).$$

This follows from (a) and (b).

For $n \geq 0$, set $L_n = (u'_n u'_{n-1} \cdots u'_1)^b L$. We show by induction on $n \geq 0$ that

$$(d) \quad u_n w_n g_D L_n = L_n,$$

$$(e) \quad (Lx, 1) \cdot h_{J,y,D} \subset G_\Delta(L_n u_n w_n, 1) \cdot h_{J,y,D}.$$

For $n = 0$, $u_0 w_0 g_D L_0 = {}^{ba\delta(b)g_D} L = {}^{b(xg_D)} L = L_0$ and

$$\begin{aligned} (Lx, 1) \cdot h_{J,y,D} &= (La, b^{-1}) \cdot h_{J,y,D} \subset G_\Delta(bLa, 1) \cdot h_{J,y,D} \\ &= G_\Delta(L_0 u_0 w_0, 1) \cdot h_{J,y,D}. \end{aligned}$$

Assume now that $n > 0$ and that (d) and (e) hold when n is replaced by $n - 1$. Then

$$\begin{aligned} u_n w_n g_D L_n &= u'_n w'_n \delta(u'_n) g_D L_{n-1} = u'_n (u_{n-1} w_{n-1} g_D L_{n-1}) = u'_n L_{n-1} = L_n, \\ (Lx, 1) \cdot h_{J,y,D} &\subset G_\Delta(L_{n-1} w'_n, (u'_n)^{-1}) \cdot h_{J,y,D} = G_\Delta(u'_n L_{n-1} w'_n, 1) \cdot h_{J,y,D} \\ &= G_\Delta(L_n u_n w_n, 1) \cdot h_{J,y,D}. \end{aligned}$$

Thus (d) and (e) are proved.

By (c), there exists $m > 0$ such that $w_m \in W^{\delta(J)}$, $u_m \in W_{J_m}$ and $w_m \delta(J_m) = J_m$. By [St, Lemma 7.3], $u_m = l_1 l_2 (w_m g_D) l_1^{-1} (w_m g_D)^{-1}$ for $l_1 \in L_{J_m}$ and $l_2 \in L_{J_m} \cap B$. Set $L' = l_1^{-1} L_m$. Then ${}^{l_2 w_m g_D} L' = L'$ and

$$\begin{aligned} (Lx, 1) \cdot h_{J,y,D} &\subset G_\Delta(L_m u_m w_m, 1) \cdot h_{J,y,D} = G_\Delta(L_m l_1 l_2 w_m, l_1) \cdot h_{J,y,D} \\ &= G_\Delta(L' l_2 w_m, 1) \cdot h_{J,y,D} = G_\Delta(L'_\Delta ((L' \cap B) l_2 w_m, 1) \cdot h_{J,y,D}) \\ &\subset G_\Delta(B w_m, 1) \cdot h_{J,y,D} = Z_{J,y,D;w_m}. \end{aligned}$$

Proposition 2.9. *Let σ be an involution on \hat{G} with $\sigma(g_D) = g_D^{-1}$ and $\sigma(P_J) = {}^{y^{-1}} P_{J'}$. Define the map $\sigma : Z_{J,y,D} \rightarrow Z_{J',y^{-1},D^{-1}}$ by*

$$\sigma(P, Q, \gamma) = (\sigma(P), \sigma(Q), \sigma(\gamma)).$$

Then there exists a map $\wp : W^{\delta(J)} \rightarrow W^{\delta(J)}$ such that

$$\partial \circ \sigma(Z_{J,y,D;w}) = Z_{J,y,D;\wp(w)}$$

for all $w \in W^{\delta(J)}$. In particular, $\wp \circ \wp = id$.

Proof. We have that $\partial \circ \sigma(Z_{J,y,D}) = Z_{J,y,D}$ and $\partial \circ \sigma(h_{J,y,D}) = h_{J,y,D}$. Then

$$\begin{aligned} \partial \circ \sigma(Z_{J,y,D;w}) &= \partial \circ \sigma(G_\Delta(wL_{\delta(I(J,w,\delta))}), 1) \cdot h_{J,y,\delta} \\ &= G_\Delta(\sigma(L_{\delta(I(J,w,\delta))})\sigma(w)^{-1}, 1) \cdot h_{J,y,\delta}. \end{aligned}$$

Notice that

$$\sigma(w)^{-1}g_D\sigma(L_{\delta(I(J,w,\delta))}) = \sigma(w^{-1}g_D^{-1}L_{\delta(I(J,w,\delta))}) = \sigma(L_{\delta(I(J,w,\delta))}).$$

By Lemma 2.8, there exists a map $\wp : W^{\delta(J)} \rightarrow W^{\delta(J)}$ such that such that $\partial \circ \sigma(Z_{J,y,D;w}) \subset Z_{J,y,D;\wp(w)}$.

Then $\partial \circ \sigma(Z_{J,y,D;\wp(w)}) \subset Z_{J,y,D;\wp \circ \wp(w)}$. Since $(\partial \circ \sigma) \circ (\partial \circ \sigma) = id$, $\partial \circ \sigma(Z_{J,y,D;\wp(w)}) \subset Z_{J,y,D;w}$. Therefore $\partial \circ \sigma(Z_{J,y,D;w}) = Z_{J,y,D;\wp(w)}$ and $\wp \circ \wp = id$. \square

3. THE G^σ -STABLE PIECES

3.1. From now on, we assume that the characteristic of \mathbf{k} is 0 or sufficiently large. Let σ and τ be involutions on G with $\sigma(T) = \tau(T) = T$ and $\sigma\tau(B) = B$. Set $\hat{G} = G \ltimes \langle \sigma\tau \rangle$. Then σ, τ acts on \hat{G} by

$$\begin{aligned} \sigma(g, (\sigma\tau)^n) &= (\sigma(g), (\tau\sigma)^n) = (\sigma(g), (\sigma\tau)^{-n}), \\ \tau(g, (\sigma\tau)^n) &= (\tau(g), (\sigma\tau)^{-n}). \end{aligned}$$

Let $D = (G, \sigma\tau)$. Then D is a connected component of \hat{G} and $\sigma(D) = D^{-1}$. Set $g_D = (1, \sigma\tau)$. Then $\sigma(g_D) = g_D^{-1}$. As in 1.2, we use the symbol δ for the induced maps of $\sigma\tau$ on Φ, I and W .

3.2. Let $J \subset I$ with $\tau(L_J) = L_J$. Then $\sigma(P_J), P_{\delta(J)} = \sigma\tau(P_J)$ have a common Levi. Let $J' \subset I$ and $y \in {}^{J'}W^{\delta(J)}$ be such that $\sigma(P_J) = y^{-1}P_{J'}$. By [C, 2.8.7], we have that $y\delta(J) = J'$.

Recall that $L_J^{\circ\sigma\tau} = \{l \in L_J; \tau(l) = l^{-1}\}$. Define the P_J -action on $G \times L_J^{\circ\sigma\tau}$ by $p \cdot (g, l) = (gp^{-1}, \pi_J(p)l\tau(\pi_J(p))^{-1})$. Let $G \times_{P_J} L_J^{\circ\sigma\tau}$ be the quotient space. In Proposition 3.4, we will show that $G \times_{P_J} L_J^{\circ\sigma\tau}$ can be identified with certain subvariety of $Z_{J,y,D}$. Before doing that, let us recall the following result (see [Sl, page 26, lemma 4]).

Lemma 3.3. *Let H be a closed subgroup of G and $\Phi : X \rightarrow G/H$ be a G -equivariant morphism from the G -variety X to the homogeneous space G/H . Let $E \subset X$ be the fiber $\Phi^{-1}(H)$. Then E is stabilized by H and the map $\Psi : G \times_H E \rightarrow X$ sending (g, e) to $g \cdot e$ defines an isomorphism of G -varieties.*

Now we prove the following result.

Proposition 3.4. *The map $G \times L_J^{\circ\sigma\tau} \rightarrow Z_{J,y,D}$ defined by $(g, l) \mapsto (\sigma(g), gl) \cdot h_{J,y,D}$ induces an isomorphism*

$$G \times_{P_J} L_J^{\circ\sigma\tau} \cong Z_{J,y,D}^{\circ\sigma\tau}.$$

In particular, $Z_{J,y,D}^{\iota\sigma}$ is irreducible if and only if $L_J^{\iota\sigma\tau}$ is irreducible.

Remark. This result is inspired by [Sp1, 2.3].

Proof. Define a G -action on $Z_{J,y,D}$ by $g \cdot z = (\sigma(g), g) \cdot z$. Then $Z_{J,y,D}^{\iota\sigma}$ is stable under the action. The G -equivariant morphism $Z_{J,y,D} \rightarrow G/P_J$ defined by $(P, Q, \rho) \mapsto P$ induces a G -equivariant morphism

$$\Phi : Z_{J,y,D}^{\iota\sigma} \rightarrow G/P_J.$$

If $(g', g) \cdot h_{J,y,D} \in \Phi^{-1}(P_J)$, then $g = lu$ for some $l \in L_J$ and $u \in U_{P_J}$ and $(g', g) \cdot h_{J,y,D} = (\sigma(g), \sigma(g')) \cdot h_{J,y,D} = (1, \sigma(g')\tau(l)^{-1}) \cdot h_{J,y,D}$. Thus $\sigma(g') \in P_J$. Write $\sigma(g')$ as $l'u'$ for $l' \in L_J$ and $u' \in U_{P_J}$. Then

$$\begin{aligned} (g', g) \cdot h_{J,y,D} &= (\sigma(l'), l) \cdot h_{J,y,D} = (1, l\tau(l')^{-1}) \cdot h_{J,y,D}, \\ (\sigma(g), \sigma(g')) \cdot h_{J,y,D} &= (\sigma(l), l') \cdot h_{J,y,D} = (1, l'\tau(l)^{-1}) \cdot h_{J,y,D}. \end{aligned}$$

So $l\tau(l')^{-1} = \tau(l'\tau(l)^{-1})^{-1}$ and $\Phi^{-1}(P_J) = (1, L_J^{\iota\sigma\tau}) \cdot h_{J,y,D}$. Now the proposition follows from Lemma 3.3. \square

3.5. By [Gi, Lemma 3.3.0], the “Lang map” $l \mapsto l\tau(l)^{-1}$ gives rise to a L_J -equivariant isomorphism from L_J/L_J^τ to the identity component of $L_J^{\iota\sigma\tau}$. Now define the P_J action on $G \times L_J/L_J^\tau$ by $p \cdot (g, z) = (gp^{-1}, \pi_J(p)z)$. Let

$$X_{J,\tau} = G \times_{P_J} L_J/L_J^\tau$$

be the quotient space. Then by the identification of $G \times_{P_J} L_J^{\iota\sigma\tau}$ with $Z_{J,y,D}^{\iota\sigma}$, we may identify $X_{J,\tau}$ with the irreducible component of $Z_{J,y,D}^{\iota\sigma}$ that contains $h_{J,y,D}$.

We can also naturally identify $X_{J,\tau}$ with $G/U_{P_J}L_J^\tau$. In particular, G acts transitively on $X_{J,\tau}$.

3.6. Examples. (1). We write $\mathbf{G} = G \times G$, $\mathbf{T} = T \times T$, $\mathbf{B} = B \times B$. Denote by σ the permutation involution $(g, g') \rightarrow (g', g)$ of \mathbf{G} . Let $J \subset I$ and $\mathbf{J} = (J, J)$. Then $\mathbf{X}_{\mathbf{J},\sigma} = Z_{J,1,G}$ and the \mathbf{G}^σ -action on $\mathbf{X}_{\mathbf{J},\sigma}$ is just the $G_{\Delta,1}$ -action on $Z_{J,1,G}$.

(2). Let σ be an involution on G . Then $X_{I,\sigma} = G/G^\sigma$ is a symmetric space.

Proposition 3.7. *Let θ be an involution on G such that $\theta(T) = T$. Then for each $G^\theta \times B$ -orbit v on G , there exists $x \in v$ such that $x^{-1}\theta(x) \in N(T)$.*

See [RS, Theorem 1.3] for the special case where $\theta(B) = B$ and [Sp1, 3.5] for the general case.

3.8. Let θ be an involution on G with $\theta(T) = T$. Let $J \subset I$ with $\theta(L_J) = L_J$. Define

$$\mathcal{J}_{J,\theta} = \{w \in W; g^{-1}\theta(g) \in B_J w \theta(B_J) \text{ for some } g \in L_J\}.$$

Set

$$\mathcal{W}(J, \sigma, \tau) = \{w \in W^{\delta(J)}; w\delta(u) \in \mathcal{J}_{I,\sigma} \text{ for some } u \in \mathcal{J}_{J,\tau}\}.$$

Proposition 3.9. *Let $w \in W^{\delta(J)}$. Then $Z_{J,y,D;w} \cap X_{J,\tau} \neq \emptyset$ if and only if $w \in \mathcal{W}(J, \sigma, \tau)$.*

Remark. *If $Z_{J,y,D;w} \cap X_{J,\tau} \neq \emptyset$, then $Z_{J,y,D;w} \cap Z_{J,y,D}^{\iota \circ \sigma} \neq \emptyset$. In other words, $\iota \circ \sigma(Z_{J,y,D;w}) \cap Z_{J,y,D;w} \neq \emptyset$. By Proposition 2.9, $\iota \circ \sigma(Z_{J,y,D;w}) = Z_{J,y,D;w}$.*

Proof. Let $g \in G$ and $b \in B$ with $(gbw, g) \cdot h_{J,D} \in X_{J,\tau}$. Then

$$h_{J,y,D} = (w^{-1}b^{-1}g^{-1}\sigma(g), g^{-1}\sigma(g)\sigma(b)\sigma(w)) \cdot h_{J,y,D}.$$

So $w^{-1}b^{-1}g^{-1}\sigma(g) \in \sigma(P_J)$ and $g^{-1}\sigma(g) \in Bw\sigma(P_J) = BwL_{\delta(J)}\sigma(U_{P_J})$. In other words, $g^{-1}\sigma(g) \in Bw\delta(u)\sigma(B)$ for some $u \in W_J$. By definition, $w\delta(u) \in \mathcal{I}_{I,\sigma}$.

By Proposition 3.7, we may write g as kxb' for $k \in G^\sigma$, $x \in G$ with $x^{-1}\sigma(x) \in w\delta(u)T$ and $b' \in B$. Notice that

$$\begin{aligned} (kxBw, kxB) \cdot h_{J,y,D} &= (kxw(w^{-1}B), kxB) \cdot h_{J,y,D} \\ &= (k\sigma(x)\delta(u)^{-1}(w^{-1}B), kxB) \cdot h_{J,y,D}. \end{aligned}$$

Thus $\delta(u)^{-1}(w^{-1}B), B) \cdot h_{J,y,D} \cap X_{J,\tau} \neq \emptyset$.

In other words, there exists $b_1 \in w^{-1}B$, $b_2 \in B$ and $p \in G$, such that $(\delta(u)^{-1}b_1, b_2) \cdot h_{J,y,D} = (\sigma(p), p) \cdot h_{J,y,D}$. Hence $b_2^{-1}p \in P_J$ and $p \in P_J$. Therefore $b_1 \in \delta(u)\sigma(p)\sigma(P_J) = \sigma(P_J)$. Write b_1 as $b_1 = lb'_1$ for $l \in L_{\delta(J)}$ and $b'_1 \in U_{\sigma(P_J)}$. Then it is easy to see that $l \in w^{-1}B$. Since $w \in W^{\delta(J)}$, $l \in L_{\delta(J)} \cap B$. Therefore

$$\begin{aligned} (\delta(u)^{-1}b_1, b_2) \cdot h_{J,y,D} &= (1, b_2g_D^{-1}l^{-1}\delta(u)g_D) \cdot h_{J,y,D} \in (1, Bu) \cdot h_{J,y,D} \\ (\sigma(p), p) \cdot h_{J,y,D} &= (1, \pi_J(p)\tau(\pi_J(p))^{-1}) \cdot h_{J,D}. \end{aligned}$$

Therefore, $\pi_J(p)\tau(\pi_J(p))^{-1} \in Bu\tau(B)$ and $u \in \mathcal{J}_{J,\tau}$. Hence $w \in \mathcal{W}(J, \sigma, \tau)$.

On the other hand, assume that $u \in W_J$ with $w\delta(u) \in \mathcal{I}_{I,\sigma}$ and $u \in \mathcal{J}_{J,\tau}$. Then by Proposition 3.7, there exists $x \in G$ with $x^{-1}\sigma(x) \in w\delta(u)T$ and $y \in L_J$ with $y^{-1}\tau(y) \in uT$. Then

$$\begin{aligned} (\sigma(xy^{-1}), xy^{-1}) \cdot h_{J,y,D} &= (x, x)(x^{-1}\sigma(x), y^{-1}\tau(y)) \cdot h_{J,y,D} \\ &\in (x, x)(w\delta(u)T, uT) \cdot h_{J,y,D} \\ &= (x, x)(wT, 1) \cdot h_{J,y,D} \subset Z_{J,y,D;w}. \end{aligned}$$

Thus $Z_{J,y,D;w} \cap X_{J,\tau} \neq \emptyset$. □

We can see that for $w \in \mathcal{W}(J, \sigma, \tau)$, the elements $u \in W_J$ such that $w\delta(u) \in \mathcal{I}_{I,\sigma}$ and $u \in \mathcal{I}_{J,\tau}$ may not be unique. We will study $\mathcal{W}(J, \sigma, \tau)$ in more detail and show that there exists a “distinguished” element u for each $w \in \mathcal{W}(J, \sigma, \tau)$.

Lemma 3.10. *Let $x, w \in W^{\delta(J)}$, $u \in W_J$ and $v \in W_{\delta(J)}$. If $\text{supp}(v) = \delta(\text{supp}(u))$ and $wv = ux$, then $u \in W_{I(J,w,\delta)}$.*

Proof. Let $(J_n, w_n)_{n \geq 0}$ be the element in $\mathcal{T}(J, \delta)$ that corresponds to w . By 2.2, it suffices to prove that $u \in W_{J_n}$ for $n \geq 0$.

We argue by induction on n . For $n = 0$ this is clear. Assume now that $n > 0$ and that (a) holds when n is replaced by $n - 1$. Write v as ab for $a \in W_{\delta(J_n)}$ and $b \in {}^{\delta(J_n)}W \cap W_{\delta(J_{n-1})}$ and x as cd for $c \in W_{J_{n-1}}$ and $d \in {}^{J_{n-1}}W^{\delta(J)}$. Then $w_{n-1}v = (w_{n-1}aw_{n-1}^{-1})w_{n-1}b$, where $w_{n-1}aw_{n-1}^{-1} \in W_{J_{n-1}}$ and $w_{n-1}b \in {}^{J_{n-1}}W$. Thus $wv \in W_{J_{n-1}}w_{n-1}v = W_{J_{n-1}}w_{n-1}b$ and $ux \in W_{J_{n-1}}d$. Then $w_{n-1}b = d$. Since $d, w_{n-1} \in W^{\delta(J)}$, we see that $b = 1$ and $v \in W_{\delta(J_n)}$. By our assumption, $u \in W_{J_n}$. \square

Lemma 3.11. *Let $w \in W^{\delta(J)}$ and $u \in W_J$. If $\sigma(w) = \delta(u)^{-1}w^{-1}u$, then $u\tau(\Phi_{I(J,w,\delta)}) = \Phi_{I(J,w,\delta)}$.*

Proof. Since $w\delta(I(J, w, \delta)) = I(J, w, \delta)$, there exists a bijection

$$\rho : I(J, w, \delta) \rightarrow I(J, w, \delta)$$

such that $w\alpha_{\delta(j)} = \alpha_{\rho(j)}$ for each $k \in I(J, w, \delta)$.

Applying σ on both sides, we have that $\delta(u)^{-1}w^{-1}u\sigma(\alpha_{\delta(j)}) = \sigma(\alpha_{\rho(j)})$. Hence

$$u\tau(\alpha_j) = u\sigma(\alpha_{\delta(j)}) = w\delta(u)\sigma(\alpha_{\rho(j)}) = w\delta(u\tau(\alpha_{\rho(j)})).$$

Then

$$\begin{aligned} \sum_{j \in I(J, w, \delta), u\tau(\alpha_j) > 0} u\tau(\alpha_j) &= w\delta\left(\sum_{j \in I(J, w, \delta), u\tau(\alpha_j) > 0} u\tau(\alpha_j)\right), \\ \sum_{j \in I(J, w, \delta), u\tau(\alpha_j) < 0} u\tau(\alpha_j) &= w\delta\left(\sum_{j \in I(J, w, \delta), u\tau(\alpha_j) < 0} u\tau(\alpha_j)\right). \end{aligned}$$

Assume that

$$\begin{aligned} \sum_{j \in I(J, w, \delta), u\tau(\alpha_j) > 0} u\tau(\alpha_j) &= \sum_{i \in J} a_i \alpha_i, \\ \sum_{j \in I(J, w, \delta), u\tau(\alpha_j) < 0} u\tau(\alpha_j) &= - \sum_{i \in J} b_i \alpha_i. \end{aligned}$$

Since $w \in W^{\delta(J)}$, then $w\delta\{i \in J; a_i \neq 0\} = \{i \in J; a_i \neq 0\}$ and $w\delta\{i \in J; b_i \neq 0\} = \{i \in J; b_i \neq 0\}$.

Moreover, for each $j \in I(J, w, \delta)$, $u\tau(\alpha_j)$ is a linear combination of α_i with $a_i \neq 0$ or $b_i \neq 0$. Thus the vector space spanned by $u\tau(\alpha_j)$ for $j \in I(J, w, \delta)$ is a subspace of the vector space spanned by α_i with $a_i \neq 0$ or $b_i \neq 0$ and the cardinality of $\{i \in J; a_i \neq 0, \text{ or } b_i \neq 0\}$ is larger than or equal to the cardinality of $I(J, w, \delta)$.

By the definition of $I(J, w, \delta)$, we have that $I(J, w, \delta) = \{i \in J; a_i \neq 0, \text{ or } b_i \neq 0\}$. Therefore, the vector space spanned by $u\tau(\alpha_j)$ for $j \in I(J, w, \delta)$ equals the vector space spanned by α_k for $k \in I(J, w, \delta)$. Hence $u\tau(\Phi_{I(J, w, \delta)}) = \Phi_{I(J, w, \delta)}$. The lemma is proved. \square

Lemma 3.12. *Let θ be an involution on G with $\theta(T) = T$. Let $K \subset I$, $w \in W$ with $w\theta(\Phi_K^+) = \Phi_K^+$. If $g^{-1}\theta(g) \in P_K w\theta(P_K)$. Then there exists $l \in L_K$, such that $(gl)^{-1}\theta(gl) \in Bw\theta(B)$.*

Remark. *This result is due to J. F. Thomsen by private communication.*

Proof. Set $\theta' = \text{Ad}(w) \circ \theta$. Then θ' is an automorphism on G . By our assumption, $\theta'(L_K) = L_K$ and $\theta'(B_K) = B_K$. We have that $g^{-1}\theta(g) \in U_{P_K} l_1 w\theta(U_{P_K})$ for some $l_1 \in L_K$.

By [St, Lemma 7.3], the map $L_K \times B_K \rightarrow L_K$ defined by $(l, b) \mapsto l^{-1}b\theta'(l)$ is surjective. Thus $l^{-1}l_1\theta'(l) \in B_K$ for some $l \in L_K$. Hence $(gl)^{-1}\theta(gl) \in U_{P_K} B_K w\theta(U_{P_K}) \subset Bw\theta(B)$. The lemma is proved. \square

Corollary 3.13. *For $w \in \mathcal{W}(J, \sigma, \tau)$, there exists a unique element $u \in W_J$ such that $u\tau(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+$, $w\delta(u) \in \mathcal{J}_{I, \sigma}$ and $u \in \mathcal{J}_{J, \tau}$.*

Proof. Let $a \in W_J$ with $w\delta(a) \in \mathcal{J}_{I, \sigma}$ and $a \in \mathcal{J}_{J, \tau}$. Then $\tau(a) = a^{-1}$ and $\sigma(w\delta(a)) = (w\delta(a))^{-1}$. Hence $\sigma(w) = \delta(a)^{-1}w^{-1}a$. By Lemma 3.11, $a\tau(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+$. Now write a as $a = bu$ for $b \in W_{I(J, w, \delta)}$ and $u \in W_J$ with $u\tau(\Phi_{I(J, w, \delta)}^+) \subset \Phi^+$. Then $u\tau(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+$.

By assumption, there exists $g \in G$ such that $g^{-1}\sigma(g) \in Bw\delta(a)\sigma(B)$. Notice that

$$w\delta(u)\sigma(\Phi_{I(J, w, \delta)}^+) = w\delta(u\tau(\Phi_{I(J, w, \delta)}^+)) = w\delta(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+.$$

Thus $w\sigma(a) \in W_{I(J, w, \delta)} w\delta(u)$ and $g^{-1}\sigma(g) \in P_{I(J, w, \delta)} w\delta(u)\sigma(P_{I(J, w, \delta)})$. By Lemma 3.12, there exists $l \in L_{I(J, w, \delta)}$, such that $(gl)^{-1}\sigma(gl) \in Bw\delta(u)\sigma(B)$. In particular, $w\delta(u) \in \mathcal{J}_{I, \sigma}$.

Similarly, $u \in \mathcal{J}_{J, \tau}$. The existence is proved.

Now we prove the uniqueness. Assume that $u_1, u_2 \in W_J$ with $u_1\tau(\Phi_{I(J, w, \delta)}^+) = u_2\tau(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+$ and $\sigma(w) = \delta(u_1)^{-1}w^{-1}u_1 = \delta(u_2)^{-1}w^{-1}u_2$. Set $v = u_1 u_2^{-1}$. Then $vw = w\delta(v)$. By Lemma 3.10, $v \in W_{I(J, w, \delta)}$. Notice that $u_1 = vu_2$ and $u_1\tau(\Phi_{I(J, w, \delta)}^+) = u_2\tau(\Phi_{I(J, w, \delta)}^+) = \Phi_{I(J, w, \delta)}^+$. Then $v = 1$ and $u_1 = u_2$. The corollary is proved. \square

3.14. Unless otherwise stated, we fix $w \in \mathcal{W}(J, \sigma, \tau)$ in the rest of this section. We will simply write $I(J, w, \delta)$ as K . Let u be the unique element in W_J with $u\tau(\Phi_K^+) = \Phi_K^+$, $w\delta(u) \in \mathcal{J}_{I, \sigma}$ and $u \in \mathcal{J}_{J, \tau}$. Set

$$G_w = \{g \in G; g^{-1}\sigma(g) \in P_K w\delta(u)\sigma(P_K)\},$$

$$L_w = \{l \in L_J; l^{-1}\tau(l) \in (P_K \cap L_J)u\tau(P_K \cap L_J)\}.$$

Let v_1 be a $G^\sigma \times P_K$ -orbit on G_w and v_2 be a $L_J^\tau \times (P_K \cap L_J)$ -orbit on L_w .

Notice that $w\delta(u)\sigma(\Phi_K^+) = \Phi_K^+$. By Lemma 3.12, there exists $g \in v_1$ with $g^{-1}\sigma(g) \in Bw\delta(u)B$. By Proposition 3.7, there exists $x_1 \in v_1$ with $x_1^{-1}\sigma(x_1) \in w\delta(u)T$. Similarly, there exists $x_2 \in v_2$ such that $x_2^{-1}\tau(x_2) \in uT$.

Set $\sigma' = \text{Ad}(x_1^{-1}\sigma(x_1)) \circ \sigma$ and $\tau' = \text{Ad}(x_2^{-1}\tau(x_2)) \circ \tau$. Then σ' is an involution on G , τ' is an involution on L_J and $\sigma'(L_K) = \tau'(L_K) = L_K$. Moreover, $x_1 G^{\sigma'} x_1^{-1} = G^\sigma$ and $x_2 L_K^{\tau'} x_2^{-1} = L_K^\tau$.

Set $H = P_K \cap \tau'(P_K) \cap L_J$. Then $U_H = U_{P_K} \cap \tau'(U_{P_K}) \cap L_J$.

Proposition 3.15. *Keep the notation of 3.14. Then*

$$\pi_J(G^{\sigma'} \cap P_K) L_K L_J^{\tau'} = H L_J^{\tau'}.$$

The proof will be given in 3.20. The key point is to show that certain equations on U have common solutions (see Lemma 3.19). We use the exponential map $\exp : \text{Lie}(U) \rightarrow U$ to reduce this problem to the problem of solving certain equations on $\text{Lie}(U)$. (Here we use the fact that \exp is an isomorphism when the characteristic of \mathbf{k} is 0 or sufficiently large.)

The equations on $\text{Lie}(U)$ that we need to solve are nonlinear equations. We will use “linear approximation” to solve these equations. We will provide the setting for “linear approximation” in Lemma 3.16 and we will prove the existence of common solutions for the linear equations in Lemma 3.17.

Lemma 3.16. *Set*

$$u_0 = \text{Lie}(U_H) = (\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J)) \cap \tau'(\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J)).$$

Define $u_n = [u_0, u_{n-1}]$ for $n > 0$ and $u'_n = u_n \oplus \text{Lie}(U_{P_J})$ for $n \geq 0$. Then

- (1) $u_{n+1} \subset u_n$ and $u'_{n+1} \subset u'_n$ for all $n \geq 0$.
- (2) $[u'_0, u'_n] \subset u'_{n+1}$ for all $n \geq 0$.
- (3) $u_n = \{0\}$ and $u'_n = \text{Lie}(U_{P_J})$ for $n \gg 0$.
- (4) $\tau'(u_n) = u_n$ and $\sigma'(u_n) \subset u'_n$ for $n \geq 0$.

Proof. We prove (1) by induction on $n \geq 0$. We have

$$[\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J), \text{Lie}(U_{P_{I(J,w,\delta)}}) \cap \text{Lie}(L_J)] \subset \text{Lie}(U_{P_K}) \cap \text{Lie}(L_J).$$

Thus $u_1 = [u_0, u_0] \subset \text{Lie}(U_{P_K}) \cap \text{Lie}(L_J)$.

Similarly, $u_1 \subset \tau'(\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J))$. Hence $u_1 \subset u_0$. Assume now that $n > 0$ and $u_n \subset u_{n-1}$. Then $u_{n+1} = [u_0, u_n] \subset [u_0, u_{n-1}] = u_n$. From definition, we deduce that $u'_{n+1} \subset u'_n$.

We prove (2). By definition,

$$u'_{n+1} = u_{n+1} \oplus \text{Lie}(U_{P_J}) = [u_0, u_n] \oplus \text{Lie}(U_{P_J}) = [u'_0, u'_n] + \text{Lie}(U_{P_J}).$$

(3) follows easily from definition.

We prove (4) by induction on $n \geq 0$.

By definition,

$$\begin{aligned} u_0 &= \bigoplus_{\alpha, \tau'(\alpha) \in \Phi_J^+ - \Phi_K^+} u_\alpha, \\ u'_0 &= \left(\bigoplus_{\alpha, \tau'(\alpha) \in \Phi_J^+ - \Phi_K^+} u_\alpha \right) \bigoplus \left(\bigoplus_{\alpha \in \Phi^+ - \Phi_J^+} u_\alpha \right). \end{aligned}$$

Hence $\tau'(u_0) = u_0$. Let α be a root with $\alpha, \tau'(\alpha) \in \Phi_J^+ - \Phi_K^+$, then

$$\sigma'(\alpha) = w\delta(u)\sigma(\alpha) = w\delta\tau'(\alpha) \in w\delta(\Phi_J^+ - \Phi_K^+) \subset \Phi^+ - \Phi_K^+.$$

If moreover, $\sigma'(\alpha) \in \Phi_J^+$, then $\delta^{-1}(w^{-1}\alpha) = \tau'\sigma'(\alpha) \in \Phi_J$. Since $w \in W^{\delta(J)}$ and $\alpha \in \Phi^+$, we deduce that $\tau'\sigma'(\alpha) \in \Phi_J^+$. Therefore $\sigma'(u_0) \subset u'_0$.

Assume now that $n > 0$ and that (4) holds when n is replaced by $n - 1$. Then

$$\begin{aligned}\tau'(u_n) &= \tau'([u_0, u_{n-1}]) = [\tau'(u_0), \tau'(u_{n-1})] = [u_0, u_{n-1}] = u_n, \\ \sigma'(u_n) &= \sigma'([u_0, u_{n-1}]) = [\sigma'(u_0), \sigma'(u_{n-1})] \subset [u'_0, u'_{n-1}] \subset u'_n.\end{aligned}$$

(4) is proved. \square

Lemma 3.17. *Keep the notation of Lemma 3.16. Let $x \in L_K$ with $\sigma'(x) = x^{-1}$. Set $\tilde{\sigma}' = \text{Ad}(x) \circ \sigma'$. Then for each $a \in u_n$ with $\tau'(a) = -a$, there exists $b \in u'_n$ such that $\tilde{\sigma}'(b) = b$ and $\pi_J(b) - \tau'(\pi_J(b)) = a$.*

Proof. We show that

(a) For any $n \geq 0$, $\text{Ad}(x)u'_n = u'_n$.

We use induction on n . Notice that $x \in L_K$ and $\tau'(L_K) = L_K$. Then

$$\begin{aligned}\text{Ad}(x)(\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J)) &= \text{Lie}(U_{P_{I(J,w,\delta)}}) \cap \text{Lie}(L_J), \\ \text{Ad}(x)\tau'(\text{Lie}(U_{P_{I(J,w,\delta)}}) \cap \text{Lie}(L_J)) &= \tau'(\text{Lie}(U_{P_K}) \cap \text{Lie}(L_J)), \\ \text{Ad}(x)\text{Lie}(U_{P_J}) &= \text{Lie}(U_{P_J}).\end{aligned}$$

Thus $\text{Ad}(x)u'_0 = u'_0$. Assume now that $n > 0$ and that (a) hold when n is replaced by $n - 1$. Then

$$\text{Ad}(x)u'_n = \text{Ad}(x)[u'_0, u'_{n-1}] + \text{Ad}(x)\text{Lie}(U_{P_J}) = [u'_0, u'_{n-1}] + \text{Lie}(U_{P_J}) = u'_n.$$

(a) is proved.

(b) For any $n \geq 0$, $\tilde{\sigma}'(u_n) \subset u'_n$.

This follows from (a) and Lemma 3.16(4).

(c) Let α be a root. If $(w\delta)^n\alpha \in \Phi_J^+$ for all $n \geq 0$, then $\alpha \in \Phi_K^+$.

We have that $(w\delta)^k\alpha = \alpha$ for some $k > 0$. Then $w\delta(\sum_{i=1}^{k-1}(w\delta)^i\alpha) = \sum_{i=1}^{k-1}(w\delta)^i\alpha$. Write $\sum_{i=1}^{k-1}(w\delta)^i\alpha$ as $\sum_{j \in J} a_j \alpha_j$. Then

$$w\delta\{j \in J; a_j \neq 0\} = \{j \in J; a_j \neq 0\}.$$

Hence if $a_j \neq 0$, then $j \in I(J, w, \delta)$. Therefore $\alpha \in \Phi_K^+$.

(c) is proved.

(d) $(\pi_J \tilde{\sigma}' \tau')^n \text{Lie}(U_{P_K}) = \{0\}$ for $n \gg 0$.

Notice that $\pi_J \tilde{\sigma}' \tau' \in \pi_J \text{Ad}(L_K) \text{Ad}(wg_D)$. Thus

$$(\pi_J \tilde{\sigma}' \tau')^n \text{Lie}(U_{P_K}) = (\pi_J \text{Ad}(wg_D))^n \text{Lie}(U_{P_K}).$$

Now (d) follows from (c).

Now set

$$b = -\frac{1 + \tilde{\sigma}'}{2} \tau' \sum_{i \geq 0} (\pi_J \tilde{\sigma}' \tau')^i a.$$

By (d), b is well-defined. By (b) and Lemma 3.16(4), $b \in u'_n$. Since $\tilde{\sigma}'$ is an involution, we have that $\tilde{\sigma}'(b) = b$. Now

$$\begin{aligned} (1 - \tau')\pi_J(b) &= \sum_{i \geq 0} (\pi_J \tilde{\sigma}' \tau')^i a/2 - \tau' \sum_{i \geq 0} (\pi_J \tilde{\sigma}' \tau')^i a/2 \\ &\quad + \tau' \sum_{i \geq 1} (\pi_J \tilde{\sigma}' \tau')^i a/2 - \sum_{i \geq 1} (\pi_J \tilde{\sigma}' \tau')^i a/2 \\ &= a/2 - \tau'(a/2) = a \end{aligned}$$

□

3.18. Let us recall the Campbell-Hausdorff formula.

Let $\exp : \text{Lie}(U) \rightarrow U$ be the exponential map. Then for $X, Y \in \text{Lie}(U)$, $\exp(X)\exp(Y) = \exp(X + Y + \sum_{n \geq 1} f_n(X, Y))$, where

$$f_n(X, Y) = \sum_{\substack{r_i + t_i > 0; \\ r_1 + t_1 + \dots + r_s + t_s = n-1}} a_{r_1, t_1, \dots, r_s, t_s} (adX)^{r_1} (adY)^{t_1} \dots (adX)^{r_s} (adY)^{t_s} Y$$

and the coefficients $a_{r_1, t_1, \dots, r_s, t_s}$ only depends on $r_1, t_1, \dots, r_s, t_s$.

Lemma 3.19. *Keep the notation of Lemma 3.16 and 3.17. Let $a \in u_0$ with $\tau'(a) = -a$. Then there exists $b \in u'_0$ such that $\tilde{\sigma}'(b) = b$ and $\exp(\pi_J(b))\exp(-\tau'(\pi_J(b))) = \exp(a)$.*

Proof. For $b \in \text{Lie}(P_J)$, we simply write $\pi_J(b)$ as \bar{b} . It suffices to prove the following statement: for each n , there exists $b_n \in u'_0$, such that

$$\tilde{\sigma}'(b_n) = b_n \text{ and } \exp(\bar{b}_n)\exp(-\tau'(\bar{b}_n)) \in \exp(a + u_n).$$

We prove by induction on n . For $n = 0$, we may choose $b_0 = 0$. Assume that $n > 0$ and that $\exp(\bar{b}_{n-1})\exp(-\tau'(\bar{b}_{n-1})) \in \exp(a + u_{n-1})$ for $\tilde{\sigma}'(b_{n-1}) = b_{n-1} \in u'_0$. By Campbell-Hausdorff formula,

$$\bar{b}_{n-1} - \tau'(\bar{b}_{n-1}) + \sum_{i=2}^{n-1} f_i(\bar{b}_{n-1}, -\tau'(\bar{b}_{n-1})) \in a + u_{n-1}.$$

Thus $\bar{b}_{n-1} - \tau'(\bar{b}_{n-1}) + \sum_{i=2}^n f_i(\bar{b}_{n-1}, -\tau'(\bar{b}_{n-1})) \in a + a_{n-1} + u_n$ for some $a_{n-1} \in u_{n-1}$.

Since $\tau'(\exp(\bar{b})\exp(-\tau'(\bar{b}))) = (\exp(\bar{b})\exp(-\tau'(\bar{b})))^{-1}$, we have that $\tau'(a_{n-1}) = -a_{n-1}$. By the definition of f_i and Lemma 3.16(4),

$$\begin{aligned} f_i(\bar{b}_{n-1} + u_{n-1}, -\tau'(\bar{b}_{n-1} + u_{n-1})) &= f_i(\bar{b}_{n-1} + u_{n-1}, -\tau'(\bar{b}_{n-1}) + u_{n-1}) \\ &\subset f_i(\bar{b}_{n-1}, -\tau'(\bar{b}_{n-1})) + [u_0, u_{n-1}] \\ &= f_i(\bar{b}_{n-1}, -\tau'(\bar{b}_{n-1})) + u_n. \end{aligned}$$

By Lemma 3.17, there exists $b'_{n-1} \in u'_{n-1}$, such that $\tilde{\sigma}'(b'_{n-1}) = b'_{n-1}$ and $b'_{n-1} - \tau'(b'_{n-1}) \in a_{n-1} + \text{Lie}(U_{P_J})$. Set $b_n = b_{n-1} - b'_{n-1}$. Then

$$\bar{b}_n - \tau'(\bar{b}_n) + \sum_{i=2}^n f_i(\bar{b}_n, -\tau'(\bar{b}_n)) \in a + u_n.$$

In other words, $\exp(\bar{b}_n)\exp(-\tau'(\bar{b}_n)) \in \exp(a + u_n)$. \square

3.20. The proof of Proposition 3.15. By definition

$$\pi_J(\text{Lie}(U_{P_K}) \cap \sigma' \text{Lie}(U_{P_K})) \subset \bigoplus_{\alpha, \sigma'(\alpha) \in \Phi_J^+ - \Phi_K^+} u_\alpha.$$

Let α be a root with $\alpha, \sigma'(\alpha) \in \Phi_J^+ - \Phi_K^+$. It is easy to see that $\tau'(\alpha) \in \Phi_J \cap \tau' \sigma'(\Phi_J^+) = \Phi_J \cap \delta^{-1} w^{-1}(\Phi_J^+)$. Since $w \in W^{\delta(J)}$, we deduce that $\tau'(\alpha) \in \Phi_J^+$. Notice that $\tau'(\Phi_K) = \Phi_K$. Thus $\tau'(\alpha) \in \Phi_J^+ - \Phi_K^+$. Hence $\tau' \pi_J(\text{Lie}(U_{P_K}) \cap \sigma' \text{Lie}(U_{P_K})) \subset \text{Lie}(U_{P_K})$. Therefore

$$\tau' \pi_J(P_K \cap \sigma'(P_K)) = \tau'(L_K) \tau' \pi_J(P_K \cap \sigma'(P_K)) \subset P_K.$$

In other words, $\pi_J(P_K \cap \sigma'(P_K)) \subset \tau'(P_K)$. Therefore

$$\pi_J(P_K \cap \sigma'(P_K)) \subset H.$$

In particular, $\pi_J(G^{\sigma'} \cap P_K) L_K L_J^{\tau'} \subset H L_J^{\tau'}$.

Let $l \in L_K$ and $p \in U_H$. Then $p \tau'(p)^{-1} = \exp(a)$ for some $a \in \text{Lie}(U_H)$ with $\tau'(a) = -a$. Set $\tilde{\sigma}' = \text{Ad}(l \sigma'(l)^{-1}) \circ \sigma'$. By Lemma 3.19, there exists $g \in G^{\tilde{\sigma}'} \cap U_{P_K}$ such that $\pi_J(g) \tau'(\pi_J(g))^{-1} = \exp(a)$. So $p \in \pi_J(G^{\tilde{\sigma}'} \cap U_{P_K}) L_J^{\tau'}$. Thus $lp \in l \pi_J(l^{-1} G^{\sigma'} l \cap U_{P_K}) L_J^{\tau'} = l \pi_J(G^{\sigma'} \cap U_{P_K}) L_J^{\tau'}$. The proposition is proved.

3.21. Keep the notation of 3.14. Set

$$X_{J,\tau;v_1,v_2} = x_1 G^{\sigma'} L_K U_{P_J} L_J^{\tau'} x_2^{-1} / U_{P_J} L_J^{\tau} = G^{\sigma} x_1 L_K x_2^{-1} U_{P_J} L_J^{\tau} / U_{P_J} L_J^{\tau}.$$

We call $X_{J,\tau;v_1,v_2}$ a G^{σ} -stable piece in $X_{J,\tau}$.

Lemma 3.22. *The variety $X_{J,\tau;v_1,v_2}$ consists of the element $gl^{-1} U_{P_J} L_J^{\tau}$, where $g \in v_1$ with $g^{-1} \sigma(g) \in w \delta(u) T$ and $l \in v_2$ with $l^{-1} \tau(l) \in (P_K \cap L_J) u$. In particular, $X_{J,\tau;v_1,v_2}$ is independent of the choice of x_1 and x_2 .*

Proof. It is easy to see that $g^{-1} \sigma(g) \in w \delta(u) T$ for $g \in x_1 G^{\sigma'} = G^{\sigma} x_1$ and $l^{-1} \tau(l) \in L_K u$ for $l \in x_2 L_K$.

Let $g = k x_1 p$ for $k \in G^{\sigma}$ and $p \in P_K$. If moreover, $g^{-1} \sigma(g) \in w \delta(u) T$, then $\sigma'(p) \in P_K$. By 3.20, $\pi_J(p) \in H$ and $p \in H U_{P_J}$.

Let $l = k' x_2 p'$ for $k' \in L_J^{\tau}$ and $p' \in P_K \cap L_J$. If moreover, $l^{-1} \tau(l) \in (P_K \cap L_J) u$, then $(p')^{-1} \tau'(p') \in P_K \cap L_J$ and $\tau'(p') \in P_K \cap L_J$. Thus $p' \in H$.

By Proposition 3.15, $gl^{-1} \in G^{\sigma} x_1 H U_{P_J} H x_2^{-1} L_J^{\tau} = G^{\sigma} x_1 L_K x_2^{-1} U_{P_J} L_J^{\tau}$. Hence $gl^{-1} U_{P_J} L_J^{\tau} \in X_{J,\tau;v_1,v_2}$. \square

Theorem 3.23. *We have that*

$$X_{J,\tau} = \bigsqcup_{w \in \mathcal{W}(J,\sigma,\tau)} \bigsqcup_{v_1 \in G^{\sigma} \backslash G_w / P_K} \bigsqcup_{v_2 \in L_J^{\tau} \backslash L_w / (P_K \cap L_J)} X_{J,\tau;v_1,v_2}.$$

Proof. By Proposition 3.9, $X_{J,\tau} = \bigsqcup_{w \in \mathcal{W}(J,\sigma,\tau)} (X_{J,\tau} \cap Z_{J,y,D;w})$.

Let $w \in \mathcal{W}(J,\sigma,\tau)$, $v_1 \in G^\sigma \setminus G_w/P_K$ and $v_2 \in L_J^\tau \setminus L_w/(P_K \cap L_J)$. Then

$$\begin{aligned} X_{J,\tau;v_1,v_2} &\subset (G^\sigma)_\Delta(\sigma(x_1 L_K x_2^{-1}), x_1 L_K x_2^{-1}) \cdot h_{J,y,D} \\ &= (G^\sigma)_\Delta(x_1 w \delta(u), x_1 L_K u L_K) \cdot h_{J,y,D} \\ &= (G^\sigma)_\Delta(x_1 w, x_1 L_K) \cdot h_{J,y,D} \subset Z_{J,y,D;w}. \end{aligned}$$

Let $z \in Z_{J,y,D;w} \cap X_{J,\tau}$. By the proof of Proposition 3.9, z can be written as $z = (gbw, g) \cdot h_{J,D}$, where $b \in B$, $g \in G$ with $g^{-1}\sigma(g) \in Bw\delta(u')B$ for some $u' \in \mathcal{J}_{J,\tau}$. By the proof of Corollary 3.13, $u' \in W_K u$. Thus $g^{-1}\sigma(g) \in P_K w \delta(u) \sigma(P_K)$. In other words, $g \in v_1$ for some $v_1 \in G^\sigma \setminus G_w/P_K$. Now

$$\begin{aligned} z &\in ((G^\sigma)_\Delta(x_1 P_K w, x_1 P_K) \cdot h_{J,y,D}) \cap X_{J,\tau} \\ &= (G^\sigma)_\Delta((\sigma(x_1) \delta(u))^{-1} w^{-1} P_K w, x_1 P_K) \cdot h_{J,y,D} \cap X_{J,\tau}. \end{aligned}$$

Similar to the proof of Proposition 3.9, $z = (\sigma(kx_1 l^{-1}), kx_1 l^{-1}) \cdot h_{J,y,D}$, where $k \in G^\sigma$ and $l \in v_2$ with $l^{-1}\tau(l) \in (P_K \cap L_J)u$. Thus $l \in v_2$ for some $v_2 \in L_J^\tau \setminus L_w/(P_K \cap L_J)$. By Lemma 3.22, $z \in X_{J,t;v_1,v_2}$.

Hence $X_{J,\tau} = \bigsqcup_{w \in \mathcal{W}(J,\sigma,\tau)} \bigcup_{v_1 \in G^\sigma \setminus G_w/P_K} \bigcup_{v_2 \in L_J^\tau \setminus L_w/(P_K \cap L_J)} X_{J,\tau;v_1,v_2}$.

Let $w \in \mathcal{W}(J,\sigma,\tau)$, $v_1, v'_1 \in G^\sigma \setminus G_w/P_K$ and $v_2, v'_2 \in L_J^\tau \setminus L_w/(P_K \cap L_J)$. Assume that $X_{J,\tau;v_1,v_2} \cap X_{J,\tau;v'_1,v'_2} \neq \emptyset$, i.e., there exists $g_1 \in v_1, g_2 \in v'_1$, $l_1 \in v_2, l_2 \in v'_2$ such that $g_1^{-1}\sigma(g_1), g_2^{-1}\sigma(g_2) \in w\delta(u)T$, $l_1^{-1}\tau(l_1), l_2^{-1}\tau(l_2) \in (P_K \cap L_J)u$ and

$$(\sigma(g_1 l_1^{-1}), g_1 l_1^{-1}) \cdot h_{J,y,D} = (\sigma(g_2 l_2^{-1}), g_2 l_2^{-1}) \cdot h_{J,y,D}.$$

Notice that

$$\begin{aligned} (\sigma(g_1 l_1^{-1}), g_1 l_1^{-1}) \cdot h_{J,y,D} &\in (g_1 w, g_1 P_K) \cdot h_{J,y,D}; \\ (\sigma(g_2 l_2^{-1}), g_2 l_2^{-1}) \cdot h_{J,y,D} &\in (g_2 w, g_2 P_K) \cdot h_{J,y,D}. \end{aligned}$$

By 1.3 (2), $g_1 \in g_2 P_K$. In particular, $v_1 = v'_1$. By the proof of Lemma 3.22, $g_1, g_2 \in G^\sigma x_1 H U_{P_J}$. Thus

$$g_1^{-1} g_2 \in H U_{P_J} G^{\sigma'} H U_{P_J} \cap P_K = H U_{P_J} (G^{\sigma'} \cap P_K) H U_{P_J} = H U_{P_J}.$$

Now

$$\begin{aligned} (1, l_1^{-1} \tau(l_1)) \cdot h_{J,y,D} &= (1, \pi_J(g_1^{-1} g_2) l_2^{-1} \tau(l_2) \tau(\pi_J(g_1^{-1} g_2))^{-1}) \cdot h_{J,y,D} \\ &= (1, (l_2 \pi_J(g_2^{-1} g_1))^{-1} \tau(l_2 \pi_J(g_2^{-1} g_1))) \cdot h_{J,y,D}. \end{aligned}$$

Notice that $l_1, l_2 \pi_J(g_2^{-1} g_1) \in L_J$. Then

$$l_1^{-1} \tau(l_1) = (l_2 \pi_J(g_2^{-1} g_1))^{-1} \tau(l_2 \pi_J(g_2^{-1} g_1)).$$

Hence $l_1 \in l_2 \pi_J(g_2^{-1} g_1) L_J^\tau$ and $v_2 = v'_2$. \square

3.24. By the proof of Theorem 3.23, each G^σ -stable piece in $X_{J,\tau}$ is an irreducible component of the intersection of $X_{J,\tau}$ with some G -stable piece in $Z_{J,y,D}$. In particular, in the Example 3.6 (1), the G^σ -stable pieces in $\mathbf{X}_{J,\sigma}$ are just the G -stable pieces in $Z_{J,1,G}$ (see remark of Proposition 3.9).

3.25. For any variety X with the action of $G^{\sigma'} \cap P_K$ (resp. $L_K^{\sigma'}$), we denote by $\text{ind}^1(X)$ (resp. $\text{ind}^2(X)$) the quotient of $G^{\sigma'} \times X$ modulo the action of $G^{\sigma'} \cap P_K$ (resp. $L_K^{\sigma'}$) defined by $g \cdot (g', x) = (g'g^{-1}, g \cdot x)$.

Recall that $\pi_J(G^{\sigma'} \cap P_K) \subset H$ (see 3.20). Unless otherwise stated, the action of $G^{\sigma'} \cap P_K$ on $HL_J^{\tau'}$ is defined by $g \cdot h = \pi_J(g)h$ and the action of $G^{\sigma'} \cap P_K$ on L_K is defined by $g \cdot l = \pi_K(g)l$.

Lemma 3.26. *The map $G^{\sigma'} \times HL_J^{\tau'} \rightarrow G$ defined by $(g, l) \mapsto x_1 g l x_2^{-1}$ induces an isomorphism $\kappa : \text{ind}^1(HL_J^{\tau'} / L_J^{\tau'}) \xrightarrow{\cong} X_{J,\tau;v_1,v_2}$.*

Proof. By 1.3 (2), the map $\pi : Z_{J,y,D;w} \rightarrow G/P_K$ defined by

$$\pi((gpw, g) \cdot h_{J,y,D}) = gP_K$$

for $g \in G$ and $p \in P_K$ is well-defined and is a G -equivariant morphism. Then its restriction to $(x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2}$ is a $G^{\sigma'}$ -equivariant morphism.

Notice that $(x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2} \subset (G^{\sigma'})_\Delta(P_K w, P_K) \cdot h_{J,y,D}$. Then

$$\pi((x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2}) \subset G^{\sigma'} P_K / P_K \cong G^{\sigma'} / G^{\sigma'} \cap P_K.$$

Since $G^{\sigma'}$ acts transitively on $G^{\sigma'} / G^{\sigma'} \cap P_K$, we have that

$$\pi((x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2}) = G^{\sigma'} / G^{\sigma'} \cap P_K.$$

By the proof of Theorem 3.23,

$$\begin{aligned} & (\pi|_{(x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2}})^{-1}(G^{\sigma'} \cap P_K) = (P_K w, P_K) \cdot h_{J,y,D} \cap (x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2} \\ & = \{(w, l^{-1}\tau(l)) \cdot h_{J,y,D}; l \in v_2 \text{ with } l^{-1}\tau(l) \in (P_K \cap L_J)u\}. \end{aligned}$$

We have shown in the proof of Lemma 3.22 that $l \in v_2$ with $l^{-1}\tau(l) \in (P_K \cap L_J)u$ if and only if $l \in x_2^{-1}L_J^{\tau'}H$. Thus

$$(\pi|_{(x_1^{-1}, x_1^{-1})X_{J,\tau;v_1,v_2}})^{-1}(G^{\sigma'} \cap P_K) \cong HL_J^{\tau'} / L_J^{\tau'}.$$

Now the lemma follows from Lemma 3.3. \square

Lemma 3.27. *Keep the notation of 3.14. Then*

- (1) $U_H / U_H^{\tau'}$ is an affine space.
- (2) The map $L_K \times U_H \rightarrow H$ defined by $(l, p) \mapsto lp$ induces an isomorphism $L_K / L_K^{\tau'} \times U_H / U_H^{\tau'} \rightarrow HL_J^{\tau'} / L_J^{\tau'}$.

Proof. (1) Notice that τ' is an order-2 linear endomorphism on $\text{Lie}(U_H)$. So $\text{Lie}(U_H) = \text{Lie}(U_H)^{\tau'} \oplus \text{Lie}(U_H)^{1-\tau'}$ and $\text{Lie}(U_H) / \text{Lie}(U_H)^{\tau'}$ is an affine space.

Moreover, the isomorphism $\exp : \text{Lie}(U_H) \rightarrow U_H$ induces an isomorphism $\text{Lie}(U_H) / \text{Lie}(U_H)^{\tau'} \rightarrow U_H / U_H^{\tau'}$. Part (1) is proved.

Part (2) is obvious. \square

The following result is an easy consequence of the above lemma and Proposition 3.15.

Corollary 3.28. *The projection map $HL_J^{\tau'}/L_J^{\tau'} \rightarrow L_K/L_K^{\tau'}$ induces an affine space bundle map*

$$\vartheta : \text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}) \rightarrow \text{ind}^1(L_K/L_K^{\tau'}).$$

Moreover, this map induces a bijection from the set of $G^{\sigma'}$ -orbits on $\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$ to the set of $G^{\sigma'}$ -orbits on $\text{ind}^1(L_K/L_K^{\tau'})$.

Notice that the map κ induces a bijection between the set of G^{σ} -orbits on $X_{J,\tau;v_1,v_2}$ and the set of $G^{\sigma'}$ -orbits on $\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$. Then

Corollary 3.29. *There is a bijection between the set of G^{σ} -orbits on $X_{J,\tau;v_1,v_2}$ and the set of $L_K^{\sigma'}$ -orbits on $L_K/L_K^{\tau'}$.*

3.30. In the rest of this section, we assume that G is adjoint. We assume furthermore that for $\alpha \in \Phi^+$, either $\tau(\alpha) = \alpha$ or $\tau(\alpha) \in -\Phi^+$. Let I be the set of simple roots and I_0 be the set of simple roots that are fixed by τ .

Let $\overline{G/G^{\tau}}$ be the De Concini-Procesi compactification of G/G^{τ} . Then $\overline{G/G^{\tau}}$ is a smooth, projective variety that contains G/G^{τ} as an open subvariety. The G^{σ} -action on G/G^{τ} extends in a unique way to a G^{σ} -action on $\overline{G/G^{\tau}}$. Moreover,

$$\overline{G/G^{\tau}} = \bigsqcup_{I_0 \subset J \subset I; \tau(\Phi_J) = \Phi_J} \bar{X}_{J,\tau}.$$

Here $\bar{X}_{J,\tau}$ is the quotient space $G \times_{P_J} (G_J/G_J^{\tau})$, where $G_J = L_J/Z^0(L_J)$ and P_J acts on G on the right and acts on G_J/G_J^{τ} via the quotient L_J (see [Sp1, 1.4]).

Let $p_J : X_{J,\tau} \rightarrow \bar{X}_{J,\tau}$ be the projection map. Set $\bar{X}_{J,\tau;v_1,v_2} = p_J(X_{J,\tau;v_1,v_2})$. We call $\bar{X}_{J,\tau;v_1,v_2}$ a G^{σ} -stable piece in $\overline{G/G^{\tau}}$.

Theorem 3.31. *We have that*

$$\overline{G/G^{\tau}} = \bigsqcup_{I_0 \subset J \subset I; \tau(\Phi_J) = \Phi_J} \bigsqcup_{w \in \mathcal{W}(J,\sigma,\tau)} \bigsqcup_{\substack{v_1 \in G^{\sigma} \backslash G_w / P_{I(J,w,\delta)}; \\ v_2 \in L_J^{\tau} \backslash L_w / (P_{I(J,w,\delta)} \cap L_J)}} \bar{X}_{J,\sigma;v_1,v_2}.$$

Let $I_0 \subset J \subset I$ with $\tau(\Phi_J) = \Phi_J$, $w \in \mathcal{W}(J,\sigma,\tau)$, $v_1 \in G^{\sigma} \backslash G_w / P_{I(J,w,\delta)}$ and $v_2 \in L_J^{\tau} \backslash L_w / (P_{I(J,w,\delta)} \cap L_J)$. We use the same notation as in 3.14. Then $\bar{X}_{J,\tau;v_1,v_2}$ is an affine space bundle over $G^{\sigma'} \times_{G^{\sigma'} \cap P_K} L_K/L_K^{\tau'} Z^0(L_J)$. Moreover, this map induces a bijection between the set of G^{σ} -orbits on $\bar{X}_{J,\tau;v_1,v_2}$ and the set of $(L_K/Z^0(L_J))^{\sigma'}$ -orbits on $(L_K/Z^0(L_J))^{\tau'}$.

4. THE CHARACTER SHEAVES

4.1. We follow the notation of [BBD] and [BL]. Let X be an algebraic variety over \mathbf{k} and l be a fixed prime number invertible in \mathbf{k} . We write $\mathcal{D}(X)$ instead of $\mathcal{D}_c^b(X, \bar{\mathbb{Q}}_l)$. If $f : X \rightarrow Y$ is a smooth morphism with connected fibres of dimension d , then we set $\tilde{f}(C) = f^*(C)[d]$ for any perverse sheaf C on Y .

Let K be an algebraic group defined over \mathbf{k} . If K acts on X , we denote by $\mathcal{D}_K(X)$ the equivariant derived category of X .

4.2. Let $C_i \in \mathcal{D}_K(X)$ for $i = 1, 2, \dots, n$. For $C \in \mathcal{D}_K(X)$, we write $C \in \langle C_i; i = 1, 2, \dots, n \rangle$ if there exist $m > n$ and $C_{n+1}, \dots, C_m \in \mathcal{D}_K(X)$ such that $C_m = C$ and for each $n+1 \leq i \leq m$, there exists $1 \leq j, k < i$ such that (C_j, C_i, C_k) is a distinguished triangle in $\mathcal{D}_K(X)$.

If X, C, C_i are as above and $Y \xrightarrow{f} X \xrightarrow{g} Z$ are K -equivariant morphisms, then

- (a) $f^*(C) \in \langle f^*(C_i); i = 1, 2, \dots, n \rangle,$
- (b) $g_!(C) \in \langle g_!(C_i); i = 1, 2, \dots, n \rangle.$

If $X = \sqcup_{1 \leq i \leq n} X_i$ is a partition of X into locally closed K -stable subvarieties such that $\sqcup_{1 \leq i \leq k} X_i$ is closed in X for any $1 \leq k \leq n$. We denote by $j_i : X_i \rightarrow X$ the inclusion maps. Let $C \in \mathcal{D}_K(X)$. Then

- (c) $C \in \langle (j_i)_!(j_i)^*(C); i = 1, 2, \dots, n \rangle.$

In the case when K is a trivial group, the notation above is slightly different from the one defined in [L2, 32.15]. Namely, $C \in \langle C_i; i = 1, 2, \dots, n \rangle$ if and only if there exists a sequence $\{C'_i; i \in \mathbf{Z}\}$ of objects in $\mathcal{D}(X)$ such that $C'_i \in \{0, C_1, C_2, \dots, C_n\}$ for all i , $C'_i = 0$ for all but finitely many i and $C \simeq \{C'_i; i \in \mathbf{Z}\}$.

4.3. Let T be a torus. Let $\mathcal{K}(T)$ be the set of isomorphism classes of Kummer local systems on T , i. e., the set of isomorphism classes of $\bar{\mathbb{Q}}_l$ -local systems \mathcal{L} of rank one on T , such that $\mathcal{L}^{\otimes m} \cong \bar{\mathbb{Q}}_l$ for some integer $m \geq 1$ invertible in \mathbf{k} .

Let X be a variety with free T -action $a : T \times X \rightarrow X$. For $\mathcal{L} \in \mathcal{K}(T)$, we denote by $\mathcal{D}^{\mathcal{L}}(X)$ the full subcategory of $\mathcal{D}(X)$ with objects $A \in \mathcal{D}(X)$ such that $a^*A \cong \mathcal{L} \boxtimes A$. If moreover, we have an action of an algebraic group K on X that commutes with the action of T , we denote by $\mathcal{D}_K^{\mathcal{L}}(X)$ the full subcategory of $\mathcal{D}_K(X)$ with objects $A \in \mathcal{D}(X)$ such that the image of A in $\mathcal{D}(X)$ is in $\mathcal{D}^{\mathcal{L}}(X)$.

4.4. Let $J \subset I$. Denote by Y_J the quotient of $G/U \times L_J/U_J$ modulo the diagonal T -action on the right. Consider the diagram

$$G/U_{P_J} \xleftarrow{p_J} G/U_{P_J} \times L_J/B_J \xrightarrow{q_J} Y_J$$

where p_J is the projection and $q_J(g, l) = (glU, lU_J)T$. Then p_J is proper and q_J is a smooth morphism with fibres isomorphic to U_J .

Define the $G^\sigma \times L_J^\tau$ -action on G/U_{P_J} by $(g, l) \cdot g' = gg'l^{-1}$, on $G/U_{P_J} \times L_J/U_J$ by $(g, l) \cdot (g', l') = (gg'l^{-1}, ll')$ and on Y_J by $(g, l) \cdot (g'U, l'U_J)T = (gg'U, ll'U_J)T$. Then the maps p_J and q_J are $G^\sigma \times L_J^\tau$ -equivariant.

Define $CH_J = (p_J)_!(q_J)^* : \mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J) \rightarrow \mathcal{D}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$ and $HC_J = (q_J)_*(p_J)^! : \mathcal{D}_{G^\sigma \times L_J^\tau}(G/U_{P_J}) \rightarrow \mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J)$.

It is easy to see that the functor CH_J is the left adjoint of HC_J .

In the case when $J = I$, CH_I is just the character functor defined in [Gi, 8.4] and HC_I is just the Harish-Chandra functor defined in *loc.cit.*

We will call CH_J a (parabolic) character functor and HC_J a (parabolic) Harish-Chandra functor.

Proposition 4.5. *Let $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$. Then A is a direct summand of $CH_J \circ HC_J(A)$.*

Remark. *The argument is inspired by [Gi, 8.5.1] and [MV, 3.6].*

Proof. Set $Z = \{(u, y); u \in L_J, y \in L_J/B_J, u \in {}^yU_J\}$. The second projection $pr_2 : Z \rightarrow L_J/B_J$ is a fibration with fibres U_J . The first projection $pr_1 : Z \rightarrow L_J$ is the ‘‘Springer resolution’’ of the unipotent variety of L_J .

Consider the following commuting diagram

$$\begin{array}{ccccc}
 G/U_{P_J} \times L_J/B_J & \xleftarrow{id \times pr_2} & G/U_{P_J} \times Z & \xrightarrow{id \times pr_1} & (G/U_{P_J}) \times L_J \\
 \swarrow p_J & & \downarrow q_J & & \downarrow h \\
 G/U_{P_J} & & Y_J & \xleftarrow{q_J} & G/U_{P_J} \times L_J/B_J \xrightarrow{p_J} G/U_{P_J} \\
 & & & & \downarrow m
 \end{array}$$

where $h(g, u, y) = (gu, y)$ and $m(g, l) = gl$.

It is easy to see that the square (h', q_J, h, q_J) is Cartesian. Now

$$\begin{aligned}
 CH_J \circ HC_J(A) &= (p_J)_*(q_J)^!(q_J)_*(p_J)^!(A)[-2d](-d) \\
 &= (p_J)_*h_*(id \times pr_2)^!(p_J)^!(A)[-2d](-d) \\
 &= m_*(id \times pr_1)_*(A \boxtimes \bar{\mathbb{Q}}_l[2d](d)) \\
 &= m_*(A \boxtimes (pr_1)_*\bar{\mathbb{Q}}_l[2d](d))
 \end{aligned}$$

Here $d = \dim(U_J)$ and $(-d)$, (d) are Tate twists.

It is known that $(pr_1)_*\bar{\mathbb{Q}}_l[2d](d)$ is a semisimple perverse sheaf on L_J and the skyscraper sheaf $(\bar{\mathbb{Q}}_l)_e$ at the identity point of L_J is a direct summand of $(pr_1)_*\bar{\mathbb{Q}}_l[2 \dim(U_J)]$. Hence $A = m_*(A \boxtimes (\bar{\mathbb{Q}}_l)_e)$ is a direct summand of $CH_J \circ HC_J(A)$. \square

Proposition 4.6. *We define the action of T on Y_J by $t(gU, lU_J)T = (gtU, lU_J)T$. Let $\mathcal{L} \in \mathcal{K}(T)$ and $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{\mathcal{L}}(Y_J)$. Then*

$$HC_J \circ CH_J(A) \in \sum_{w \in W_J} \mathcal{D}_{G^\sigma \times L_J^\tau}^{w* \mathcal{L}}(Y_J).$$

Remark. *This result is inspired by [Gr, Proposition 1.2].*

Proof. Set $Z = G/U_{P_J} \times L_J/B_J \times L_J/B_J$. Consider the following diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow a & & \searrow b & \\
 G/U_{P_J} \times L_J/B_J & & & & G/U_{P_J} \times L_J/B_J \\
 \swarrow q_J & & \searrow p_J & & \swarrow p_J \quad \searrow q_J \\
 Y_J & & G/U_{P_J} & & Y_J
 \end{array}$$

where $a(g, l, l') = (g, l)$ and $b(g, l, l') = (g, l')$.

It is easy to see that the square (a, p_J, b, p_J) is Cartesian. Notice that p_J is proper, then

$$\begin{aligned}
 HC_J \circ CH_J(A) &= (q_J)_*(p_J)^!(p_J)_!(q_J)^* = (q_J)_*a_*b^!(q_J)^!(A) \\
 &= (q_J \circ a)_*(q_J \circ b)^!(A).
 \end{aligned}$$

Now we have a partition $Z = \sqcup_{w \in W_J} Z_w$, where

$$Z_w = \{(gU_{P_J}, l_1B_J, l_2B_J); l_1, l_2 \in L_J, l_1^{-1}l_2 \in B_JwB_J\}.$$

By 4.2(c), $HC_J \circ CH_J(A) \in \langle ((q_J \circ a)|_{Z_w})_*((q_J \circ b)|_{Z_w})^!(A); w \in W_J \rangle$.

Set

$$\begin{aligned}
 Z'_w &= \{(xU, yU_J)T, (aU, bU_J)T\} \in Y_J \times Y_J; \\
 U_Jx^{-1}aU_J &= U_Jy^{-1}bU_J \subset B_JwB_J.
 \end{aligned}$$

Define the map $\pi_w : Z_w \rightarrow Z'_w$ by

$$(gU_{P_J}, l_1B_J, l_2B_J) \mapsto ((gl_1U, l_1U_J)T, (gl_2U, l_2U_J)T).$$

Then it is easy to see that π_w is an affine space bundle map with fibres isomorphic to $U_J \cap {}^wU_J$.

Let $p_w : Z'_w \rightarrow Y_J$ be the projection to the first factor and $p'_w : Z'_w \rightarrow Y_J$ be the projection to the second factor. Then $(q_J \circ a)|_{Z_w} = p_w \circ \pi_w$ and $(q_J \circ b)|_{Z_w} = p'_w \circ \pi_w$. Now

$$\begin{aligned}
 ((q_J \circ a)|_{Z_w})_*((q_J \circ b)|_{Z_w})^!(A) &= (p_w)_*(\pi_w)_*(\pi_w)^!(p'_w)^!(A) \\
 &= (p_w)_*(p'_w)^!(A)[2d](d).
 \end{aligned}$$

Here $d = \dim(U_J \cap {}^wU_J)$ and (d) is Tate twist.

Define the T -action on Z'_w by

$$t \cdot ((xU, yU_J)T, (aU, bU_J)T) = ((xwtw^{-1}U, yU_J)T, (atU, bU_J)T).$$

Then p'_w is T -equivariant and p_w is T -equivariant with respect to the twisted T -action on Y_J defined by $t \cdot (xU, yU_J)T = (xwtw^{-1}U, yU_J)T$. Thus $(p_w)_*(p'_w)^!(A) \in \mathcal{D}_{G^\sigma \times L_J^\sigma}^{w^{-1}\mathcal{L}}(Y_J)$. \square

4.7. Let $\mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$ be the full subcategory of $\mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J)$ with objects in $\bigoplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^\sigma \times L_J^\tau}^{\mathcal{L}}(Y_J)$. By [BL, 5.3], $CH_J(A)$ is semisimple for simple perverse sheaf $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$. Let $\mathcal{C}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$ be the set of (isomorphism classes) of simple perverse sheaves that are a constituent of $CH_J(A)$ for some $A \in \mathcal{C}_{G^\sigma \times L_J^\tau}(Y_J)$. The elements in $\mathcal{C}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$ are called (parabolic) character sheaves on G/U_{P_J} . Let $\mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$ be the full subcategory of $\mathcal{D}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$ consisting of objects whose perverse constituents are contained in $\mathcal{C}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$. Then it is easy to see that $CH_J(\mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)) \subset \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$.

Similar to [L3, Proposition 6.7(a)], we have the following result.

Proposition 4.8. *Let $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$. Then $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$ if and only if $HC_J(A) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$.*

Proof. If $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$, then by Proposition 4.6, $HC_J(A) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$. Conversely, assume that $HC_J(A) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$. Then $CH_J \circ HC_J(A) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$. Hence by Proposition 4.5, $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$. \square

4.9. Let $\pi : G/U_{P_J} \rightarrow X_{J,\tau} = G/U_{P_J}L_J^\tau$ be the quotient map. We call a simple perverse sheaf A in $\mathcal{D}_{G^\sigma}(X_{J,\tau})$ a character sheaf on $X_{J,\tau}$ if $\tilde{\pi}(A) \in \mathcal{C}_{G^\sigma \times L_J^\tau}(G/U_{P_J})$. They form a set $\mathcal{C}_{G^\sigma}(X_{J,\tau})$.

Let $\mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$ be the full subcategory of $\mathcal{D}_{G^\sigma}(X_{J,\tau})$ that corresponds to $\mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(G/U_{P_J})$ under π^* . Then $\mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$ is the full subcategory of $\mathcal{D}_{G^\sigma}(X_{J,\tau})$ consisting of objects whose perverse constituents are contained in $\mathcal{C}_{G^\sigma}(X_{J,\tau})$.

Now we describe $\mathcal{C}_{G^\sigma}(X_{J,\tau})$ and $\mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$ in a slightly different way.

4.10. Define the action of B on $G \times L_J$ by $b \cdot (g, l) = (gb^{-1}, \pi_J(b)l)$. Let $G \times_B L_J$ be the quotient spaces. Then we may identify $G \times_B L_J$ with $G/U_{P_J} \times L_J/B_J$ via $(g, l) \mapsto (glU_{P_J}, l^{-1}B_J)$. Under this identification, the map $p_J : G \times_B L_J \rightarrow G/U_{P_J}$ sends (g, l) to glU_{P_J} and the map $q_J : G \times_B L_J \rightarrow Y_J$ sends (g, l) to $(glU, l^{-1}U_J)T$.

Consider the following diagram

$$\begin{array}{ccccc} G/U_{P_J} & \xleftarrow{p_J} & G \times_B L_J & \xrightarrow{q_J} & Y_J \\ \pi \downarrow & & \downarrow i_J & & \\ X_{J,\tau} & \xleftarrow{p'_J} & G \times_B L_J/L_J^\tau & & \end{array}$$

where i_J is the projection and $p'_J(g, l) = glU_{P_J}L_J^\tau/U_{P_J}L_J^\tau$.

It is easy to see that the square (p_J, π, i_J, p'_J) is Cartesian.

Define (parabolic) character functor $ch_J : \mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J) \rightarrow \mathcal{D}_{G^\sigma}(X_{J,\tau})$ and (parabolic) Harish-Chandra functor $hc_J : \mathcal{D}_{G^\sigma}(X_{J,\tau}) \rightarrow \mathcal{D}_{G^\sigma}(Y_J)$ as follows:

For $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J)$, $q_J^* A \in \mathcal{D}_{G^\sigma \times L_J^\tau}(G \times_B L_J)$. Let A' be the unique element in $\mathcal{D}_{G^\sigma}(G \times_B L_J/L_J^\tau)$ with $\tilde{q}_J(A) = \tilde{i}_J(A')$. Set $ch_J(A) = (p'_J)_!(A')$.

For $B \in \mathcal{D}_{G^\sigma}(X_{J,\tau})$, set $hc_J(B) = (q_J)_!(p'_J \circ i_J)^*(B)$.

Using the above diagram, one can easily see that

(a) a simple perverse sheaf in $\mathcal{D}_{G^\sigma}(X_{J,\tau})$ is a character sheaf if and only if it is a direct summand of $ch_J(A)$ for some simple perverse sheaf $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$;

(b) $A \in \mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$ if and only if $hc_J(A) \in \bigoplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^\sigma \times L_J^\tau}^{\mathcal{L}}(Y_J)$.

4.11. Notice that each $G^\sigma \times L_J^\tau$ -stable subvariety of Y_J that is also stable under the action of T defined in Position 4.6 is of the form $\sqcup_i (G_i/U \times L_i/U_J)/T$, where G_i are some $G^\sigma \times B$ -orbits on G and L_i are some $L^\tau \times B_J$ -orbits on L_J . Let $\pi : G \times L_J \rightarrow Y_J$ be the map defined by $\pi(g, l) = (gU, l^{-1}U_J)T$. Then one can show that a simple perverse sheaf in $\mathcal{D}_{G^\sigma \times L_J^\tau}(Y_J)$ is contained in $\mathcal{D}_{G^\sigma \times L_J^\tau}^{\mathcal{L}}(Y_J)$ if and only if its image under π^* is of the form $A \boxtimes B$, where A is a simple perverse sheaf in $\mathcal{D}_{G^\sigma}(G)$, that is equivariant for the right U -action and has weight \mathcal{L} for the right T -action and B is a simple perverse sheaf in $\mathcal{D}_{L_J^\tau}(L_J)$, that is equivariant for the left U -action and has weight \mathcal{L}^{-1} for the left T -action.

Let $pr : G \times L_J/L_J^\tau \rightarrow G \times_B L_J/L_J^\tau$ be the projection map. Then a simple perverse sheaf in $\mathcal{D}_{G^\sigma}(X_{J,\tau})$ is a character sheaf if and only if it is a direct summand of $(p'_J)_!(C)$, where $C \in \mathcal{D}_{G^\sigma}(G \times_B L_J/L_J^\tau)$ with $\tilde{p}r(C) = A \boxtimes B'$ for some simple perverse sheaf $A \in \mathcal{D}_{G^\sigma}(G)$, that is equivariant for the right U -action and has weight \mathcal{L} for the right T -action and some simple perverse sheaf $B \in \mathcal{D}(L_J/L_J^\tau)$, that is equivariant for the left U -action and has weight \mathcal{L}^{-1} for the left T -action.

4.12. Recall that we have the G^σ -stable pieces decomposition

$$X_{J,\tau} = \bigsqcup_{w \in \mathcal{W}(J, \sigma, \tau)} \bigsqcup_{v_1 \in G^\sigma \backslash G_w / P_K} \bigsqcup_{v_2 \in L_J^\tau \backslash L_w / (P_K \cap L_J)} X_{J,\tau; v_1, v_2}.$$

Now we define the character sheaves on each piece $X_{J,\tau; v_1, v_2}$. The definition is similar to [L3, 4.6].

We keep the notation of 3.14. Consider the diagram

$$L_K/L_K^{\tau'} \xleftarrow{a_1} G^{\sigma'} \times L_K/L_K^{\tau'} \xrightarrow{a_2} ind^1(L_K/L_K^{\tau'}) \xleftarrow{\vartheta} ind^1(HL_J^{\tau'}/L_J^{\tau'})$$

where a_1 and a_2 are projections.

Notice that $G^{\sigma'} \cap P_K = (G^{\sigma'} \cap U_{P_K}) \rtimes L_K^{\sigma'}$, $G^{\sigma'} \cap U_{P_K}$ is unipotent and $G^{\sigma'} \cap U_{P_K}$ acts trivially on $L_K/L_K^{\tau'}$. By [MV, A6],

$$(a) \quad \mathcal{D}_{G^{\sigma'} \cap P_K}(L_K/L_K^{\tau'}) = \mathcal{D}_{L_K^{\sigma'}}(L_K/L_K^{\tau'}).$$

Define the action of $G^{\sigma'} \times (G^{\sigma'} \cap P_K)$ on $G^{\sigma'} \times L_K/L_K^{\tau'}$ by $(g, p) \cdot (g', l) = (gg'p^{-1}, \pi_K(p)l)$. Then $G^{\sigma'}$ acts freely on $G^{\sigma'} \times L_K/L_K^{\tau'}$ and $L_K/L_K^{\tau'}$ is the quotient space. By [BL, Theorem 2.6.2],

$$(b) \quad a_1^* : \mathcal{D}_{G^{\sigma'} \cap P_K}(L_K/L_K^{\tau'}) \rightarrow \mathcal{D}_{G^{\sigma'} \times (G^{\sigma'} \cap P_K)}(G^{\sigma'} \times L_K/L_K^{\tau'})$$

is an equivalence of categories.

Similarly, $G^{\sigma'} \cap P_K$ acts freely on $G^{\sigma'} \times L_K/L_K^{\tau'}$ and $\text{ind}^1(L_K/L_K^{\tau'})$ is the quotient space. By [BL, Theorem 2.6.2],

$$(c) \quad a_2^* : \mathcal{D}_{G^{\sigma'}}(Z) \rightarrow \mathcal{D}_{G^{\sigma'} \times (G^{\sigma'} \cap P_K)}(G^{\sigma'} \times L_K/L_K^{\tau'}) \text{ is an}$$

equivalence of categories.

Lemma 4.13. *The functors*

$$\begin{aligned} \vartheta^* : \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(L_K/L_K^{\tau'})) &\rightarrow \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})) \\ \vartheta_! : \mathcal{D}_{G^{\sigma'}}^{cs}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})) &\rightarrow \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(L_K/L_K^{\tau'})) \end{aligned}$$

are equivalences of categories.

Proof. By Corollary 3.28, ϑ is an affine space bundle map. Hence $\vartheta_! \vartheta^*(C)$ is just a shift of C for $C \in \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(L_K/L_K^{\tau'}))$.

Again by Corollary 3.28, for any $C' \in \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$, C' is constant along each fiber of ϑ . Hence $\vartheta^* \vartheta_!(C')$ is also a shift of C' . The lemma is proved. \square

4.14. Combining the above lemma with 4.12 (a), (b) and (c), we have that the categories $\mathcal{D}_{L_K^{\sigma'}}(L_K/L_K^{\tau'})$ and $\mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ are naturally equivalent.

For $X \in \mathcal{D}_{L_K^{\sigma'}}^{cs}(L_K/L_K^{\tau'})$, let X' be the unique element in $\mathcal{D}_{G^{\sigma'}}(Z)$ such that $a_2^* X' = a_1^* X$. Set $\tilde{X} = \vartheta^*(X')$.

As in 4.7, we denote by $\mathcal{D}_{G^{\sigma'}}^{cs}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ the full subcategory of $\mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ whose objects are \tilde{X} as above. The simple perverse sheaves that are contained in $\mathcal{D}_{G^{\sigma'}}^{cs}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ are called character sheaves on $\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$.

By Lemma 3.26, $\kappa : \text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}) \xrightarrow{\cong} X_{J,\tau;v_1,v_2}$ is an isomorphism. Hence $\kappa^* : \mathcal{D}_{G^{\sigma}}(X_{J,\tau;v_1,v_2}) \rightarrow \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ is an equivalence of categories. A simple perverse sheaf C in $\mathcal{D}_{G^{\sigma}}(X_{J,\tau;v_1,v_2})$ is called a character sheaf on $X_{J,\tau;v_1,v_2}$ if $\kappa^*(C)$ is a character sheaf on $\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$. We also denote by $\mathcal{D}_{G^{\sigma}}^{cs}(X_{J,\tau;v_1,v_2})$ the subcategory of $\mathcal{D}_{G^{\sigma}}(X_{J,\tau;v_1,v_2})$ that corresponds to $\mathcal{D}_{G^{\sigma'}}^{cs}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ under κ^* .

Now we describe $\mathcal{D}_{G^{\sigma'}}^{cs}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$ in a different way. This description will be used to prove the main theorem.

4.15. Consider the diagram

$$Y \xleftarrow{a} Z_1 \xrightarrow{b} Z_2 \xrightarrow{c} \text{ind}^2(HL_J^{\tau'}/L_J^{\tau'}) \xrightarrow{d} \text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$$

where Y is the quotient of $\text{ind}^2(L_K)/U_K \times L_J^{\tau'}H/U_K$ modulo the diagonal T -action on the right, $Z_1 = \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'}$, $Z_2 = \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'}/L_J^{\tau'}$, a, b, c are analogous to q_J, i_J, p'_J in 4.10 and d is the projection map.

As in 4.10, we define

$$\begin{aligned} \tilde{c}h : \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}(Y) &\rightarrow \mathcal{D}_{G^{\sigma'}}(\text{ind}^2(HL_J^{\tau'}/L_J^{\tau'})), \\ \tilde{h}c : \mathcal{D}_{G^{\sigma'}}(\text{ind}^2(HL_J^{\tau'}/L_J^{\tau'})) &\rightarrow \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}(Y) \end{aligned}$$

as follows:

For $A \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}(Y)$, let A' be the unique element in $\mathcal{D}_{G^{\sigma'}}(Z_2)$ with $\tilde{b}(A') = \tilde{a}(A)$. Set $\tilde{c}h(A) = c_!(A')$.

For $B \in \mathcal{D}_{G^{\sigma'}}(\text{ind}^2(HL_J^{\tau'}/L_J^{\tau'}))$, set $\tilde{h}c(B) = a_!(c \circ b)^*(B)$.

Lemma 4.16. *Define the action of U_H on $L_J^{\tau'}H$ by $g \cdot g' = g'g^{-1}$. Let A be an U_H -equivariant object in $\mathcal{D}_{L_J^{\tau'}}(L_J^{\tau'}H)$ and $\pi : L_J^{\tau'}H \rightarrow L_J^{\tau'}H/U_K$ be the projection map. Then $\pi^*\pi_!(A)$ is also U_H -equivariant.*

Proof. Consider the diagram

$$\begin{array}{ccccc} L_J^{\tau'}H \times U_K \times U_H & \xrightarrow{m'} & L_J^{\tau'}H \times U_K & \xrightarrow{p_2} & L_J^{\tau'}H \\ p_1 \times id \downarrow & & \downarrow p_1 & & \downarrow \pi \\ L_J^{\tau'}H \times U_H & \xrightarrow{m} & L_J^{\tau'}H & \xrightarrow{\pi} & L_J^{\tau'}H/U_K \end{array}$$

where $m'(g, l, u) = (glu^{-1}l^{-1}, l)$, $m(g, u) = gu^{-1}$, $p_1(g, l) = gl$ and $p_2(g, l) = g$.

It is easy to see that all the squares in the above diagram are Cartesian.

We have that

$$\begin{aligned} m^*\pi^*\pi_!(A) &= m^*(p_1)_!(p_2)^*(A) = (p_1 \times id)_!(m')^*(p_2)^*(A) \\ &= (p_1 \times id)_!(m')^*(A \otimes \bar{\mathbb{Q}}_l^{U_K}). \end{aligned}$$

Consider the diagram

$$L_J^{\tau'}H \times U_K \times U_H \xrightarrow{b} L_J^{\tau'}H \times U_H \times U_K \xrightarrow{m \times id} L_J^{\tau'}H \times U_K$$

where $b(g, l, u) = (g, lul^{-1}, l)$. Then $m' = b \circ (m \times id)$.

Since A is U_H -equivariant, we have that $m^*A \cong A \otimes \bar{\mathbb{Q}}_l^{U_H}$. Hence

$$\begin{aligned} m^*\pi^*\pi_!(A) &= (p_1 \times id)_!b^*(m^*A \otimes \bar{\mathbb{Q}}_l^{U_K}) \cong (p_1 \times id)_!b^*(A \otimes \bar{\mathbb{Q}}_l^{U_H \times U_K}) \\ &= (p_1 \times id)_!(A \otimes \bar{\mathbb{Q}}_l^{U_K \times U_H}) = (p_1)_!(A \otimes \bar{\mathbb{Q}}_l^{U_K}) \otimes \bar{\mathbb{Q}}_l^{U_H} \\ &= (p_1)_!(p_2)^*(A) \otimes \bar{\mathbb{Q}}_l^{U_H} = \pi^*\pi_!(A) \otimes \bar{\mathbb{Q}}_l^{U_H}. \end{aligned}$$

□

Proposition 4.17. *Keep the notation of 4.15. Define $\pi : \text{ind}^2(L_K) \times HL_J^{\tau'} \rightarrow Y$ by $\pi(z, l) = (zU_K, l^{-1}U_K)T$. The group U_H acts on $\text{ind}^2(L_K) \times HL_J^{\tau'}$ on the second factor on the left. Then for $A \in \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}))$, $\pi^*hc(d^*A)$ is U_H -equivariant.*

Proof. Consider the following commuting diagram

$$\begin{array}{ccccc} \text{ind}^2(L_K) \times HL_J^{\tau'}/L_J^{\tau'} & \xrightarrow{a_1} & \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'}/L_J^{\tau'} & \xrightarrow{a_2} & \text{ind}^1(HL_J^{\tau'}/L_J^{\tau'}) \\ \pi_1 \downarrow & & \downarrow \pi_2 & & \downarrow \vartheta \\ \text{ind}^2(L_K) \times L_K/L_K^{\tau'} & \xrightarrow{a_3} & \text{ind}^2(L_K) \times_{B_K} L_K/L_K^{\tau'} & \xrightarrow{a_4} & \text{ind}^1(L_K/L_K^{\tau'}) \end{array}$$

where a_1, a_3 are projections, $a_2 = d \circ c$, where c, d are defined in 4.15, a_4 is analogous to a_2 and π_1, π_2 are induced from the projection map $HL_J^{\tau'}/L_J^{\tau'} \rightarrow L_K/L_K^{\tau'}$.

By Lemma 4.13, $A = \vartheta^*(B)$ for some $B \in \mathcal{D}_{G^{\sigma'}}(\text{ind}^1(L_K/L_K^{\tau'}))$. Then $a_1^*a_2^*(A) = \pi_1^*(a_4 \circ a_3)^*(B)$. Notice that π_1 is U_H -equivariant, where U_H acts on $\text{ind}^2(L_K) \times L_K/L_K^{\tau'}$ trivially. Hence $a_1^*a_2^*(A)$ is U_H -equivariant.

Consider the following commuting diagram

$$\begin{array}{ccccc} Y' & \xleftarrow{a'} & \text{ind}^2(L_K) \times HL_J^{\tau'} & \xrightarrow{b'} & \text{ind}^2(L_K) \times HL_J^{\tau'}/L_J^{\tau'} \\ a_5 \downarrow & & a_6 \downarrow & & \downarrow a_1 \\ Y & \xleftarrow{a} & \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'} & \xrightarrow{b} & \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'}/L_J^{\tau'} \end{array}$$

where $Y' = \text{ind}^2(L_K) \times L_J^{\tau'}H/U_K$, a, b are defined in 4.15, $a'(z, l) = (z, l^{-1}U_K)$, b' is analogous to b , a_5, a_6 are projection maps.

It is easy to see that $a_5 \circ a' = \pi$ and the square (a', a_5, a_6, a) is Cartesian. Now

$$\begin{aligned} \pi^*hc(d^*A) &= (a')^*a_5^*a_1^*b^*(a_2)^*(A) = (a')^*(a')_!a_6^*b^*a_2^*(A) \\ &= (a')^*(a')_!(b')^*a_1^*a_2^*(A). \end{aligned}$$

Since $a_1^*a_2^*(A)$ is U_H -equivariant, $(b')^*a_1^*a_2^*(A)$ is also U_H -equivariant. Similar to the proof of Lemma 4.16, $\pi^*hc(d^*A)$ is U_H -equivariant. □

4.18. Let $\mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$ be the full subcategory of $\mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}(Y)$ consisting of elements in $\oplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{\mathcal{L}}(Y)$ whose inverse image under the map π defined in the previous Proposition is U_H -equivariant.

Lemma 4.19. *A simple perverse sheaf in $\mathcal{D}_{G^{\sigma'}}(\text{ind}^2(HL_J^{\tau'}/L_J^{\tau'}))$ is of the form $\tilde{d}(C)$ for some character sheaf C on $\text{ind}^1(HL_J^{\tau'}/L_J^{\tau'})$ if and only if it is a direct summand of $\tilde{ch}(A)$ for some simple perverse sheaf $A \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$.*

Proof. Set $X = L_K/L_K^{\tau'}$ and $X' = HL_J^{\tau'}/L_J^{\tau'}$. Consider the following commuting diagram

$$\begin{array}{ccccccc}
L_K \times X & \xleftarrow{pr_1} & L_K \times X' & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & ind^2(L_K) \times X' \\
\pi_1 \downarrow & & \pi_2 \downarrow & & \pi_3 \downarrow & & \pi_4 \downarrow \\
L_K \times_{B_K} X & \xleftarrow{pr_2} & L_K \times_{B_K} X' & \xleftarrow{p_3} & Z' & \xrightarrow{p_4} & ind^2(L_K) \times_{B_K} X' \\
m \downarrow & & m' \downarrow & & id \times m' \downarrow & & c \downarrow \\
X & \xleftarrow{pr} & X' & \xleftarrow{p_5} & G^{\sigma'} \times X' & \xrightarrow{p_6} & ind^2(X') \\
& & pr \downarrow & & id \times pr \downarrow & & pr_3 \downarrow \\
& & X & \xleftarrow{p_7} & G^{\sigma'} \times X & \xrightarrow{p_8} & ind^2(X)
\end{array}$$

where $Z = G^{\sigma'} \times L_K \times X'$, $Z' = G^{\sigma'} \times L_K \times_{B_K} X'$, π_i, p_i are the projection maps, pr_i are induced from the projection map $pr : X' \rightarrow X$, $m'(l, l') = ll'$ and m' is analogous to m .

It is easy to see that all the squares in the above diagram are Cartesian.

Let A be a simple perverse sheaf in $\mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{\mathcal{L}}(Y)$. Similar to 4.11, $\tilde{\pi}(A) = A_1 \boxtimes A_2$, where A_1 is a simple perverse sheaf in $\mathcal{D}_{G^{\sigma'}}(ind^2(L_K))$ that is equivariant for the right U_K -action and has weight \mathcal{L} for the right T -action and A_2 is a simple perverse sheaf in $\mathcal{D}_{L_J^{\tau'}}(HL_J^{\tau'})$ that is equivariant for the left $U_K U_H$ -action and has weight \mathcal{L}^{-1} for the left T -action.

Consider the diagram

$$L_K \xleftarrow{s_1} G^{\sigma'} \times L_K \xrightarrow{s_2} ind^2(L_K)$$

where s_1, s_2 are projections. Let A'_1 be the unique element in $\mathcal{D}_{L_K^{\sigma'}}(L_K)$ with $\tilde{s}_1(A'_1) = \tilde{s}_2(A_1)$.

Since A_2 is U_H -equivariant, there exists a unique simple perverse sheaf A'_2 on X with $\tilde{s}_3 \tilde{pr}(A'_2) = A_2$, where $s_3 : HL_J^{\tau'} \rightarrow X'$ is the projection map.

Now let B be the simple perverse sheaf in $\mathcal{D}_{G^{\sigma'}}(ind^2(L_K) \times_{B_K} X')$ with $\tilde{\pi}_4(B) = A_1 \boxtimes \tilde{pr}(A'_2)$ and B' be the simple perverse sheaf in $\mathcal{D}_{G^{\sigma'}}(L_K \times_{B_K} X)$ with $\tilde{\pi}_1(B') = A'_1 \boxtimes A'_2$. Then $\tilde{p}_4(B) = \tilde{p}_3 \tilde{pr}_2(B')$. We have that

$$\begin{aligned}
\tilde{p}_6 \tilde{ch}(A) &= \tilde{p}_6 c_1(B) = (id \times m')_! \tilde{p}_4(B) = (id \times m')_! \tilde{p}_3 \tilde{pr}_2(B') \\
&= \tilde{p}_5 m'_! \tilde{pr}_2(B') = \tilde{p}_5 \tilde{pr} \widetilde{m_!}(B') = (id \times pr) \tilde{p}_7 m_!(B').
\end{aligned}$$

Let C_1 be a simple perverse sheaf in $\mathcal{D}_{G^{\sigma'}}(ind^2(X'))$ that is a direct summand of $\tilde{ch}(A)$. Then there exists a simple perverse sheaf C'

in $\mathcal{D}_{L_K^{\sigma'}}(X)$ that is a direct summand of $m_!(B')$ such that $\tilde{p}_6(C_1) = \widetilde{(id \times pr)} \tilde{p}_7(C')$. By 4.11, C' is a character sheaf.

Let $d' : ind^2(X) \rightarrow ind^1(X)$ be the projection map. Then $d' \circ p_8 = a_2$, where a_2 is defined in 4.12. Let C_2 be the unique element in $\mathcal{D}_{G^{\sigma'}}(ind^1(X'))$ with $\tilde{a}_2(C_2) = \tilde{p}_7(C')$. Then

$$\widetilde{(id \times pr)} \tilde{p}_7(C') = \widetilde{(id \times pr)} \tilde{p}_8 d'(C_2) = \tilde{p}_6 d \tilde{\vartheta}(C_2).$$

Therefore $C_1 = \tilde{d} \tilde{\vartheta}(C_2)$. The “if” part is proved.

The “only if” part can be proved in the similar way. \square

Proposition 4.20. *Let $C \in \mathcal{D}_{G^{\sigma'}}(ind^1(X'))$. Then $C \in \mathcal{D}_{G^{\sigma'}}^{cs}(ind^1(X'))$ if and only if $\tilde{h}c(\tilde{d}(C)) \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$.*

Proof. Let Y_0 be the quotient of $L_K/U_K \times L_J^{\tau'} H/U_K$ modulo the diagonal T -action on the right. Then $Y = ind^2(Y_0)$, where $L_K^{\sigma'}$ acts on Y_0 on the first factor. We also have that $ind^2(L_K) \times_{B_K} HL_J^{\tau'} = ind^2(L_K \times_{B_K} HL_J^{\tau'})$. Now consider the following commuting diagram

$$\begin{array}{ccccc} Y_0 & \xleftarrow{q} & L_K \times_{B_K} HL_J^{\tau'} & \xrightarrow{p} & HL_J^{\tau'} \\ \pi_1 \uparrow & & \pi_2 \uparrow & & \uparrow \pi_3 \\ G^{\sigma'} \times Y_0 & \xleftarrow{id \times q} & G^{\sigma'} \times (L_K \times_{B_K} HL_J^{\tau'}) & \xrightarrow{id \times p} & G^{\sigma'} \times HL_J^{\tau'} \\ \pi_4 \downarrow & & \pi_5 \downarrow & & \downarrow \pi_6 \\ Y & \xleftarrow{a} & ind^2(L_K) \times_{B_K} HL_J^{\tau'} & \xrightarrow{c'} & ind^2(HL_J^{\tau'}) \\ & & \downarrow b & & \downarrow b' \\ & & ind^2(L_K) \times_{B_K} X' & \xrightarrow{c} & ind^2(X') \end{array}$$

where p, q are analogous to p_J, q_J defined in 4.4, π_i are the projection maps, a, b, c are defined in 4.15, c' is induced from $id \times p$, $b' : ind^2(HL_J^{\tau'}) \rightarrow ind^2(HL_J^{\tau'})/L_J^{\tau'} = ind^2(X')$ is the projection map.

It is easy to see that all the squares in the above diagram are Cartesian.

Similarly to Proposition 4.5 and 4.6, we can show that

- (1) Let $A \in \mathcal{D}_{L_K^{\sigma'} \times L_J^{\tau'}}(HL_J^{\tau'})$, then some shift of A is a direct summand of $p_! q^* q_! p^*(A)$;
- (2) Let $A \in \mathcal{D}_{L_K^{\sigma'} \times L_J^{\tau'}}^{\mathcal{L}}(Y_0)$ for some $\mathcal{L} \in \mathcal{K}(T)$, then $q_! p^* p_! q^*(A) \in \sum_{w \in W_K} \mathcal{D}_{L_K^{\sigma'} \times L_J^{\tau'}}^{w^* \mathcal{L}}(Y_0)$.

Now let C be a character sheaf on $ind^1(X')$. Then there exists $\mathcal{L} \in \mathcal{K}(T)$ and a simple perverse sheaf $A \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y) \cap \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{\mathcal{L}}(Y)$ such that $\tilde{d}(C)$ is a direct summand of $\tilde{c}h(A)$. Therefore some shift of $(b')^* \tilde{d}(C)$ is a direct summand of $c'_! a^* A$.

It is easy to see that there exists a simple perverse sheaf $A' \in \mathcal{D}_{L_K^{\sigma'} \times L_J^{\tau'}}(Y_0)$ with $\pi_1^*(A') = \pi_4^*(A)$. Then

$$\begin{aligned} \pi_3^* p_! q^*(A') &= (id \times p)_! \pi_2^* q^*(A') = (id \times p)_! (id \times q)^* \pi_1^*(A') \\ &= (id \times p)_! (id \times q)^* \pi_4^*(A) = (id \times p)_! \pi_5^* a^*(A) = \pi_6^* c'_! a^*(A). \end{aligned}$$

We can show in the same way that $\pi_1^* q_! p^* p_! q^*(A') = \pi_4^* a_!(c')^* c'_! a^*(A)$. Hence $\tilde{h}c(C) \in \sum_{w \in W_K} \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{w^* \mathcal{L}}(Y)$. By Proposition 4.17, $\tilde{h}c(C) \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$.

On the other hand, for any $C_1 \in \mathcal{D}_{G^{\sigma'}}(ind^1(X'))$, there exists $C'_1 \in \mathcal{D}_{L_K^{\sigma'} \times L_J^{\tau'}}(HL_J^{\tau'})$ with $\pi_3^*(C'_1) = \pi_6^*(b')^* \tilde{d}(C_1)$. We can also show that $\pi_3^* p_! q^* q_! p^*(C'_1) = \pi_6^* c'_! a^* a_!(c')^* (b')^* \tilde{d}(C_1) = \pi_6^* c'_! a^* \tilde{h}c(\tilde{d}(C_1))$ is a shift of $\pi_6^*(b')^* \tilde{c}h \circ \tilde{h}c(\tilde{d}(C_1))$.

Hence some shift of $\pi_6^*(b')^* \tilde{d}(C_1)$ is a direct summand of $\pi_6^*(b')^* \tilde{c}h \circ \tilde{h}c(\tilde{d}(C_1))$ and some shift of $\tilde{d}(C_1)$ is a direct summand of $\tilde{c}h \circ \tilde{h}c(\tilde{d}(C_1))$. If moreover, $\tilde{h}c(C) \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$, then by Lemma 4.19, C_1 is a character sheaf. \square

Theorem 4.21. *Let $i : X_{J,\tau;v_1,v_2} \rightarrow X_{J,\tau}$ be the inclusion map. Then*

- (1) *for any $C \in \mathcal{D}_{G^{\sigma}}^{cs}(X_{J,\tau;v_1,v_2})$, $i_!(C), i_*(C) \in \mathcal{D}_{G^{\sigma}}^{cs}(X_{J,\tau})$;*
- (2) *for any $C \in \mathcal{D}^{cs}(X_{J,\tau})$, $i^*(C) \in \mathcal{D}_{G^{\sigma}}^{cs}(X_{J,\tau;v_1,v_2})$.*

Proof. (1) Define the map $m : ind^2(L_K) \rightarrow G$ by $m(g, l) = gl$ for $g \in G^{\sigma'}$ and $l \in L_K$. Now consider the following commuting diagram

$$\begin{array}{ccccc} X_{J,\tau} & \xleftarrow{f} & ind^2(L_K) \times_{B_K} X' & \xleftarrow{b} & ind^2(L_K) \times_{B_K} HL_J^{\tau'} \\ \uparrow i_J \circ p_J & & \uparrow \pi_1 & \nearrow \pi_2 & \downarrow a \\ G/U_{P_J} \times L_J/B_J & \xleftarrow{f'} & Z & \xrightarrow{\pi_3} & Y \\ \downarrow q_J & & \nwarrow \pi_4 & & \\ Y_J & & & & \end{array}$$

where $Z = ind^2(L_K) \times_{B_K} HL_J^{\tau'} \times L_J/B_J$, $f(z, x) = x_1 m(z) x x_2^{-1} U_{P_J} L_J^{\tau'}$ for $z \in ind^2(L_K)$ and $x \in X'$, $f'(z, h, l B_J) = (x_1 m(g) h x_2^{-1} U_{P_J}, l B_J)$ for $z \in ind^2(L_K)$, $h \in HL_J^{\tau'}$ and $l \in L_J$, π_2 is the projection map, $\pi_1 = b \circ \pi_2$, $\pi_3 = a \circ \pi_2$ and $\pi_4 = q_J \circ f'$.

It is easy to see that the square $(\pi_1, f, f', i_J \circ p_J)$ is Cartesian.

By Lemma 4.19, it suffices to prove that for any $A \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$ and $B \in \mathcal{D}_{G^{\sigma'}}(ind^2(L_K) \times_{B_K} X')$ with $b^*(B) = a^*(A)$, we have that $f_!(B), f_*(B) \in \mathcal{D}_{G^{\sigma}}^{cs}(X_{J,\tau})$.

Now

$$\begin{aligned} h c_J(f_!(B)) &= (q_J)_! (i_J \circ p_J)^* f_!(B) = (q_J)_! f'_! \pi_1^*(B) = (\pi_4)_! \pi_2^* b^*(B) \\ &= (\pi_4)_! \pi_2^* a^*(A) = (\pi_4)_! (\pi_3)^*(A). \end{aligned}$$

Recall that we have a partition $L_J = \sqcup_{w \in W_J} B_J w B_J$. Moreover, $B_J w B_J = \sqcup_{i \in \mathbb{N} \cup \{0\}} L_{w,i}$, where

$$L_{w,i} = \{l \in B_J w B_J; \dim(l^{-1} U_K l \cap U_J) = i\}.$$

It is easy to see that

- (a) $L_{w,i} = \emptyset$ for $i \gg 0$;
 - (b) $L_{w,i}$ is stable under the action of B_K on the left and the action of B_J on the right;
 - (c) for any $l \in L_{w,i}$, $l^{-1} U_K l \cap U_J$ is an affine space of dimension i .
- Now set

$$Z_{w,i} = \{(g, l, l') \in Z; g \in \text{ind}^2(L_K), l \in HL_J^{\tau'}, l' \in L_J/B_J, lx_2^{-1}l' \in L_{w,i}\}.$$

By (b), $Z_{w,i}$ is well-defined. By (a), $Z = \sqcup_{w \in W_J, i \in \mathbb{N} \cup \{0\}} Z_{w,i}$ is a finite partition. Hence

$$(\pi_4)_! \pi_3^*(A) \in \langle (\pi_4|_{Z_{w,i}})_! (\pi_3|_{Z_{w,i}})^*(A); w \in W_J, i \in \mathbb{N} \cup \{0\} \rangle.$$

Set

$$\begin{aligned} Z'_{w,i} = & \{((xU_K, yU_K)T, (aU, bU_J)T) \in Y \times Y_J; \\ & U_K m(x)^{-1} x_1^{-1} a U_J = U_K y^{-1} x_2^{-1} b U_J \subset L_{w,i}\}. \end{aligned}$$

Define the map $\pi_{w,i} : Z_{w,i} \rightarrow Z'_{w,i}$ by

$$(z, l, l') \mapsto ((zU_K, l^{-1}U_K)T, (x_1 m(z) l x_2^{-1} l' U, l' U_J)T)$$

for $z \in \text{ind}^2(L_K)$, $l \in HL_J^{\tau'}$ and $l' \in L_J$.

By (c), $\pi_{w,i}$ is an affine space bundle map with fibres isomorphic to $l^{-1} U_K u \cap U_J$ for $l \in L_{w,i}$.

Let $p_{w,i} : Z'_{w,i} \rightarrow Y$ and $p'_{w,i} : Z'_{w,i} \rightarrow Y_J$ be the projection maps. Similar to the proof of Proposition 4.6,

$$(\pi_4|_{Z_{w,i}})_! (\pi_3|_{Z_{w,i}})^*(A) = (p'_{w,i})_! p_{w,i}^*(A)[2i](i) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{w^{-1}\mathcal{L}}(Y_J).$$

Hence $hc_J(f_!(B)) \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$. By Proposition 4.8, $f_!(B) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$.

We can prove in the same way that $f_*(B) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$.

(2) Consider the following commuting diagram

$$\begin{array}{ccccc} & & X_{J,\tau} & \xleftarrow{p'_J} & G \times_B L_J/L_J^\tau & \xleftarrow{i_J} & G \times_B L_J \\ & \uparrow f \circ b & & & \uparrow \pi_5 & \nearrow f' & \downarrow q_J \\ \text{ind}^2(L_K) \times_{B_K} HL_J^{\tau'} & \xleftarrow{\pi_2} & Z & \xrightarrow{\pi_4} & Y_J \\ & \downarrow a & \nwarrow \pi_3 & & & & \end{array}$$

where Z, π_2, π_3, π_4 are defined above, p'_J, i_J, q_J are defined in 4.10 and $\pi_5 = i_J \circ f'$.

It is easy to see that the square $(\pi_5, p'_J, \pi_2, f \circ b)$ is Cartesian.

Let $A \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$ and $B \in \mathcal{D}_{G^\sigma}(G \times_B L_J / L_J^\tau)$ with $i_J^*(B) = q_J^*(A)$. Then

$$\begin{aligned} a_!(f \circ b)^*(p'_J)_!(B) &= a_!(\pi_2)_! \pi_5^*(B) = (\pi_3)_!(f')^*(i_J)^*(B) \\ &= (\pi_3)_!(f')^* q_J^*(A) = (\pi_3)_!(\pi_4)^*(A). \end{aligned}$$

Similarly to the proof of part (1), we can show that

$$a_!(f \circ b)^*(p'_J)_!(B) \in \bigoplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{\mathcal{L}}(Y_J).$$

By Proposition 4.17, $a_!(f \circ b)^*(p'_J)_!(B) \in \mathcal{D}_{G^{\sigma'} \times L_J^{\tau'}}^{cs}(Y)$. Now part (2) follows from Proposition 4.20. \square

Corollary 4.22. *Let C be a simple perverse sheaf in $\mathcal{D}_{G^\sigma}(X_{J,\tau})$. Then C is a character sheaf if and only if C is the perverse extension of C' , where C' is a character sheaf on some G^σ -stable piece $X_{J,\tau;v_1,v_2}$.*

Proof. Since $X_{J,\tau} = \sqcup X_{J,\tau;v_1,v_2}$, we can find a G^σ -stable piece $X_{J,\tau;v_1,v_2}$ such that $\text{supp}(C) \cap X_{J,\tau;v_1,v_2}$ is open dense in $\text{supp}(C)$. Hence $C|_{X_{J,\tau;v_1,v_2}}$ is a simple perverse sheaf on $X_{J,\tau;v_1,v_2}$. By part (2) of the above theorem, $C|_{X_{J,\tau;v_1,v_2}}$ is a character sheaf on $X_{J,\tau;v_1,v_2}$.

On the other hand, if C' is a character sheaf on $X_{J,\tau;v_1,v_2}$. By part (1) of the above theorem, $i_!(C') \in \mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$, where $i : X_{J,\tau;v_1,v_2} \rightarrow X_{J,\tau}$ is the inclusion map. Since the perverse extension of C' is a quotient of ${}^p H^0(i_!(C'))$, the perverse extension of C' is also contained in $\mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$. Therefore, the perverse extension of C' is a character sheaf on $X_{J,\tau}$. \square

5. LUSZTIG'S FUNCTORS $e_J^{J'}$ AND f_J^J

5.1. Let $J \subset J' \subset I$ with $L_J = \tau(L_J)$ and $L_{J'} = \tau(L_{J'})$.

Define the action of P_J on $G \times (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau$ by $p \cdot (g, l) = (gp^{-1}, \pi_{J'}(p)l)$. Let $Z_{J,J'}$ be the quotient space. Let $\pi : L_{J'} \cap P_J \rightarrow (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau \cong L_J$ be the projection. Then π induces a morphism $\bar{\pi} : (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau \rightarrow L_J / L_J^\tau$. The morphism $(id, \bar{\pi}) : G \times (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau \rightarrow G \times L_J / L_J^\tau$ is equivariant under the P_J -action. Then it induces a morphism $c : Z_{J,J'} \rightarrow X_{J,\tau}$.

The inclusion map $G \times (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau \rightarrow G \times L_{J'} / (L_{J'} \cap P_J)^\tau$ induces an isomorphism from $Z_{J,J'}$ to $G \times_{P_{J'}} L_{J'} / (L_{J'} \cap P_J)^\tau$. Thus the projection map $G \times_{P_{J'}} L_{J'} / (L_{J'} \cap P_J)^\tau \rightarrow X_{J',\tau}$ induces a map $d : Z_{J,J'} \rightarrow X_{J',\tau}$. We will write c as $c_{J,J'}$ and d as $d_{J,J'}$ if necessary.

Consider the diagram

$$X_{J,\tau} \xleftarrow{c} Z_{J,J'} \xrightarrow{d} X_{J',\tau}.$$

Define

$$f_{J'}^J : \mathcal{D}_{G^\sigma}(X_{J,\tau}) \rightarrow \mathcal{D}_{G^\sigma}(X_{J',\tau}), \quad e_J^{J'} : \mathcal{D}_{G^\sigma}(X_{J',\tau}) \rightarrow \mathcal{D}_{G^\sigma}(X_{J,\tau})$$

by $f_{J'}^J(A) = d_! c^*(A)$, $e_J^{J'}(A') = c_! d^*(A')$.

In the special case where $\mathbf{X}_{\mathbf{J},\sigma} = Z_{J,1,G}$ and $\mathbf{X}_{\mathbf{J}',\sigma} = Z_{J',1,G}$ (see example 3.6), the functors $e_{\mathbf{J}}^{\mathbf{J}'}$ and $f_{\mathbf{J}}^{\mathbf{J}'}$ are just the functors $e_{J'}^J$ and $f_{J'}^J$ defined in [L3, 6.1].

Proposition 5.2. *Let $J_1 \subset J_2 \subset J_3 \subset I$ with $\tau(L_{J_i}) = L_{J_i}$ for $i = 1, 2, 3$. Then $e_{J_1}^{J_2} \circ e_{J_2}^{J_3} = e_{J_1}^{J_3}$ and $f_{J_3}^{J_2} \circ f_{J_2}^{J_1} = f_{J_3}^{J_1}$.*

Remark. *This proposition is a generalization of [L3, Lemma 6.2]. The proof below is similar to the proof in loc.cit.*

Proof. Consider the following commuting diagram

$$\begin{array}{ccccc}
 & & Z_{J_1, J_3} & & \\
 & a \swarrow & & \searrow b & \\
 & Z_{J_1, J_2} & \diamond & Z_{J_2, J_3} & \\
 c_{J_1, J_2} \swarrow & & & & \searrow d_{J_2, J_3} \\
 X_{J_1, \tau} & & X_{J_2, \tau} & & X_{J_3, \tau}
 \end{array}$$

where a is defined in the similar way as the map c in 5.1 and b is induced from the projection map $G \times_{P_{J_3}} L_{J_3}/(L_{J_3} \cap P_{J_1})^\tau \rightarrow G \times_{P_{J_3}} L_{J_3}/(L_{J_3} \cap P_{J_2})^\tau$.

Notice that the square \diamond is Cartesian and $c_{J_1, J_2} \circ a = c_{J_1, J_3}$ and $d_{J_2, J_3} \circ b = d_{J_1, J_3}$. Then

$$\begin{aligned}
 e_{J_1}^{J_2} \circ e_{J_2}^{J_3} &= (c_{J_1, J_2})! (d_{J_1, J_2})^* (c_{J_2, J_3})! (d_{J_2, J_3})^* = (c_{J_1, J_2})! a! b^* (d_{J_2, J_3})^* \\
 &= (c_{J_1, J_3})! (d_{J_1, J_3})^* = e_{J_1}^{J_3}, \\
 f_{J_3}^{J_2} \circ f_{J_2}^{J_1} &= (d_{J_2, J_3})! (c_{J_2, J_3})^* (d_{J_1, J_2})! (c_{J_1, J_2})^* = (d_{J_2, J_3})! b! a^* (c_{J_1, J_2})^* \\
 &= (d_{J_1, J_3})! (c_{J_1, J_3})^* = f_{J_3}^{J_1}.
 \end{aligned}$$

□

Proposition 5.3. *We keep the notation of 5.1. Then*

- (1) *If $C \in \mathcal{D}_{G^\sigma}^{cs}(X_{J', \tau})$, then $e_{J'}^{J'}(C) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J, \tau})$.*
- (2) *If $C' \in \mathcal{D}_{G^\sigma}^{cs}(X_{J, \tau})$, then $f_{J'}^J(C') \in \mathcal{D}_{G^\sigma}^{cs}(X_{J', \tau})$.*

Remark. *The special case where $\mathbf{X}_{\mathbf{J},\sigma} = Z_{J,1,G}$ and $\mathbf{X}_{\mathbf{J}',\sigma} = Z_{J',1,G}$ was proved by Lusztig in [L3, 6.7(b)] and [L3, 6.4].*

Proof. (1) Define the action of B on $G \times (L_{J'} \cap P_J)/(L_{J'} \cap P_J)^\tau$ by $b \cdot (g, l) = (gb^{-1}, \pi_{J'}(b)l)$. Denote by $G \times_B (L_{J'} \cap P_J)/(L_{J'} \cap P_J)^\tau$ the quotient space. We also define $G \times_B (L_{J'} \cap P_J)/(L_{J'} \cap U_{P_J})^\tau$ and $G \times_B (L_{J'} \cap P_J)$ in the similar way.

Consider the following commuting diagram

$$\begin{array}{ccccc}
X_{J,\tau} & \xleftarrow{c} & Z_{J,J'} & \xrightarrow{d} & X_{J',\tau} \\
p'_J \uparrow & & f \uparrow & & p'_{J'} \uparrow \\
G \times_B L_J / L_J^\tau & \xleftarrow{h} & Z_1 & \xrightarrow{k} & G \times_B L_{J'} / L_{J'}^\tau \\
i_J \uparrow & & j \uparrow & & i_{J'} \uparrow \\
G \times_B L_J & \xleftarrow{r} & Z_3 & \xleftarrow{s} & Z_2 & \xrightarrow{t} & G \times_B L_{J'} \\
& \searrow q_J & & \downarrow v & & & \downarrow q_{J'} \\
& & & Y_J & \xrightarrow{t'} & Y_{J'}
\end{array}$$

where $Z_1 = G \times_B (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau$, $Z_2 = G \times_B (L_{J'} \cap P_J)$, $Z_3 = G \times_B (L_{J'} \cap P_J) / (L_{J'} \cap U_{P_J})^\tau$, h (resp. k) is defined in the similar way as c (resp. d), f, j, s are the projection maps, t is an inclusion and $t'((gU, lU_J)T) = (gU, lU_{J'})T$.

It is easy to see that the squares (f, c, h, p'_J) and (j, h, r, i_J) are Cartesian. Notice that we may identify Y_J with $(G/U \times (L_{J'} \cap P_J)/U_{J'})T$ in the natural way and the map t' is just the inclusion map under this identification. Thus the square $(t, q_{J'}, v, t')$ is also Cartesian.

Notice that s is an affine space bundle map. Then

$$\begin{aligned}
hc_J(e_J^{J'}(C)) &= (q_J)_!(p'_J \circ i_J)^* c_! d^*(C) = (q_J)_! r_! (f \circ j)^* d^*(C) \\
&= (q_J \circ r)_! s_! s^*(f \circ j)^* d^*(C) [2d](d) = v_! (d \circ f \circ j \circ s)^*(C) [2d](d) \\
&= v_! (p'_{J'} \circ i_{J'} \circ t)^*(C) [2d](d) = v_! t^* (p'_{J'} \circ i_{J'})^*(C) [2d](d) \\
&= (t')^* (q_{J'})_! (\bar{p}_{J'} \circ i_{J'})^*(C) [2d](d) = (t')^* hc_{J'}(C) [2d](d),
\end{aligned}$$

where $d = \dim((L_{J'} \cap U_{P_J})^\tau)$.

By Proposition 4.8 $hc_{J'}(C) \in \bigoplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^\sigma \times L_{J'}^\tau}^{\mathcal{L}}(Y_{J'})$. Therefore, $hc_J(e_J^{J'}(C)) = (t')^* hc_{J'}(C) [2d](d) \in \bigoplus_{\mathcal{L} \in \mathcal{K}(T)} \mathcal{D}_{G^\sigma \times L_J^\tau}^{\mathcal{L}}(Y_J)$. By Proposition 4.8, $e_J^{J'}(C) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J,\tau})$.

(2) By 4.10, it suffices to prove that for any simple perverse sheaf $A \in \mathcal{D}_{G^\sigma, \mathcal{L}}(G \times_B L_J / L_J^\tau)$ and $A' \in \mathcal{D}_{G^\sigma \times L_J^\tau}^{cs}(Y_J)$ with $i_J^*(A) = q_J^*(A')$, we have that $f_{J'}^J(p'_J)_!(A) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J',\tau})$.

We use the above diagram. Then

$$f_{J'}^J(p'_J)_!(A) = d_! c^*(p'_J)_!(A) = d_! f_! h^*(A) = (p'_{J'})_! k_! h^*(A).$$

Consider the following commuting diagram

$$\begin{array}{ccccc}
G \times L_J / L_J^\tau & \xleftarrow{h'} & G \times (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau & \xrightarrow{k'} & G \times L_{J'} / L_{J'}^\tau \\
\pi_1 \downarrow & & \downarrow \pi_2 & & \downarrow \pi_3 \\
G \times_B L_J / L_J^\tau & \xleftarrow{h} & G \times_B (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau & \xrightarrow{k} & G \times_B L_{J'} / L_{J'}^\tau
\end{array}$$

where π_1, π_2, π_3 are projections, $h' = (id, \bar{\pi})$ is defined in 5.1 and $k' : G \times (L_{J'} \cap P_J) / (L_{J'} \cap P_J)^\tau = G \times (L_{J'} \cap P_J) L_{J'}^\tau / L_{J'}^\tau \rightarrow G \times L_{J'} / L_{J'}^\tau$ is just the inclusion map.

It is easy to see that all the squares in the diagram are Cartesian.

Then $\pi_3^* k_! h^*(A) = k_! \pi_2^* h^*(A) = k_! (h')^* \pi_1^*(A)$. By 4.11, $\pi_1^*(A) = A_1 \boxtimes A_2$, where A_1 is a simple perverse sheaf in $\mathcal{D}_{G^\sigma}(G)$, that is equivariant for the right U -action and has weight \mathcal{L} for the right T -action and A_2 is a simple perverse sheaf in $\mathcal{D}(L_J / L_J^\tau)$, that is equivariant for the left U -action and has weight \mathcal{L}^{-1} for the left T -action. Thus $k_! (h')^* \pi_1^*(A) = A_1 \boxtimes A_3$, where $A_3 \in \mathcal{D}(L_{J'} / L_{J'}^\tau)$ is equivariant for the left U -action, has weight \mathcal{L}^{-1} for the left T -action and is supported in $(L_{J'} \cap P_J) L_{J'}^\tau / L_{J'}^\tau$. By 4.11, $f_{J'}^J(p_J)_!(A) \in \mathcal{D}_{G^\sigma}^{cs}(X_{J', \tau})$. \square

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